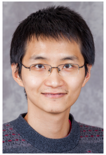
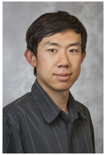
High-Dimensional Robust Mean Estimation in Nearly-Linear Time



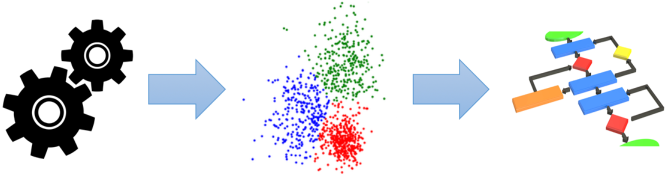


Yu Cheng [1] Ilias Diakonikolas [2] Rong Ge [1]

[1] Duke University

[2] University of Southern California

Statistical Learning



(Unknown) Parameters Samples Algorithms

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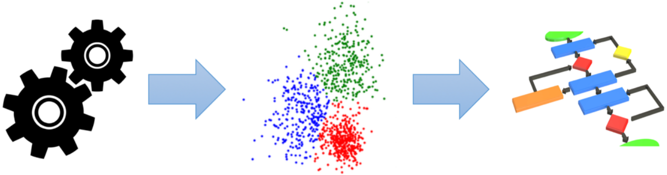

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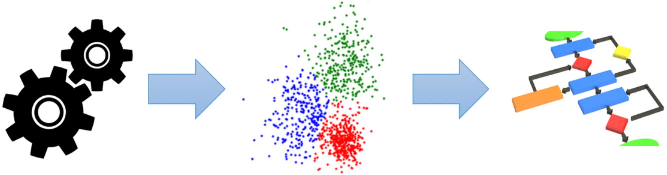

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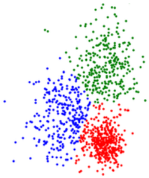

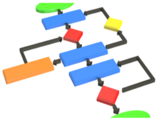

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Performance criteria:

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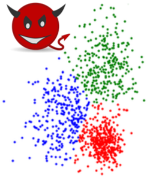
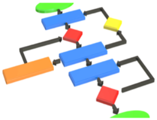
- **Robustness**

Robust Statistical Learning

(Unknown) Parameters



Corrupted samples



Algorithms

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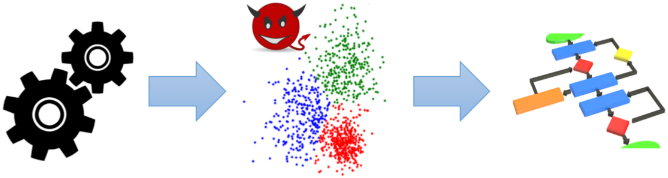

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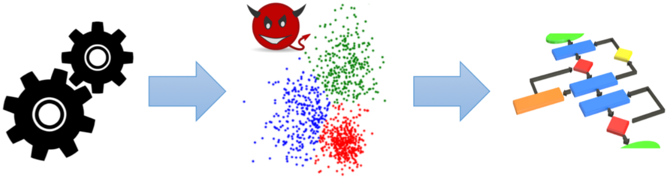

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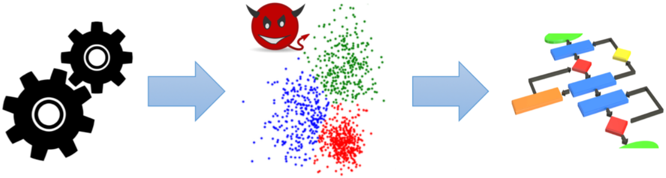

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- Data poisoning attacks, Reliable / Adversarial / Secure ML

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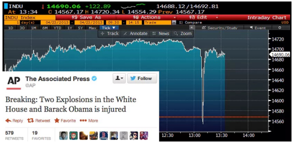

Data Poisoning: High-frequency trading algorithms.

Twitter account of the Associated Press was hacked in April 2013 (\$136 billion in 3 minutes).

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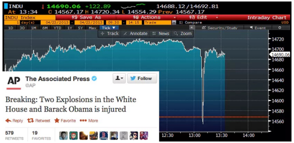

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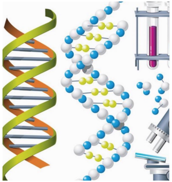

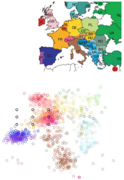

Biological Datasets: POPRES project, HGDP datasets.

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Hard to detect in several cases [Rosenberg et al., Science'02; Li et al., Science'08; Paschou et al., Medical Genetics'10]

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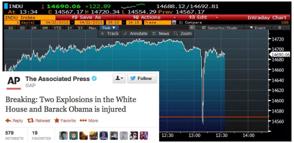

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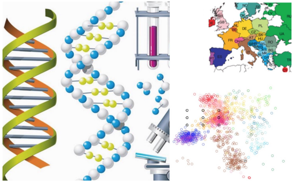

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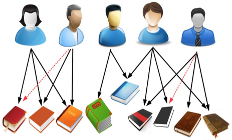

Reliable/Adversarial/Secure ML:

Recommendation Systems, Crowdsourcing, ...

Attacker can generate malicious data to maximize his objectives. [Mayzlin et al. '14] [Wang et al. '14] [Li et al. '16]

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- Running time: $O(Nd)$.

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Robust Mean Estimation

- *Input:* an ϵ-corrupted set of N samples $\{X_1, \ldots, X_N\}$ drawn from an unknown distribution D on $\mathbb{R}^d$ with mean $\mu^\star$.
- *Goal:* Learn $\mu^\star$ in ℓ_2-norm.

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Robustly learn $\mu^\star$ given ϵ-corrupted samples from $\mathcal{N}(\mu^\star, I)$:

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All these algorithms have the right sample complexity $N = O(d/\delta^2)$.

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When ϵ is constant, our algorithm has the best possible error guarantee, sample complexity, and running time (up to polylogarithmic factors).

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Robust mean estimation under bounded covariance assumptions has been used as a subroutine to obtain robust learners for a wide range of supervised learning problems that can be phrased as stochastic convex programs.

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Our result provides a faster implementation of such a subroutine, hence yields faster robust algorithms for all these problems.

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Except that ... we do not know $\mu^\star$.

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Idea: guess the mean ν and solve the SDP with parameter ν.

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We give a win-win analysis: either

- a near-optimal solution w to the primal SDP give a good answer $\widehat{\mu}_w$, or
- a near-optimal solution to the dual SDP yields a new guess ν' that is closer to $\mu^\star$ by a constant factor.

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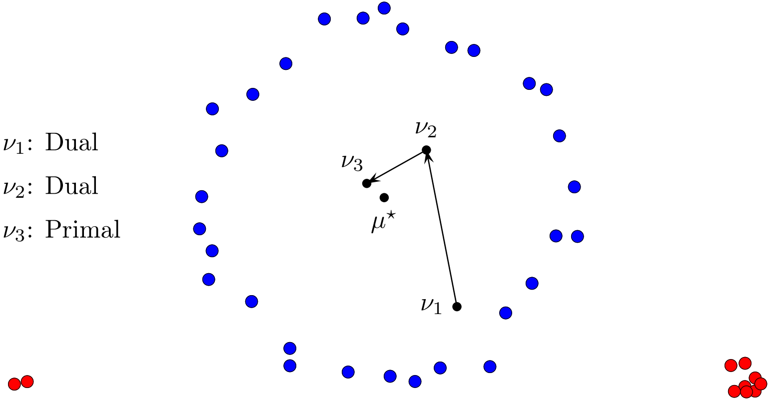
ν_1: Dual

ν_2: Dual

ν_3: Primal



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- How far is $\mu^\star$?

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- If the solution is rank-one: $M = yy^\top$, then in the direction of y, the variance is large no matter how we reweight the samples.
- Intuition: When ν is far from $\mu^\star$, y should align with $(\nu - \mu^\star)$.

Direction of $\mu^\star$: Dual SDP

Why would the dual SDP pick the direction $(\nu - \mu^\star)$?

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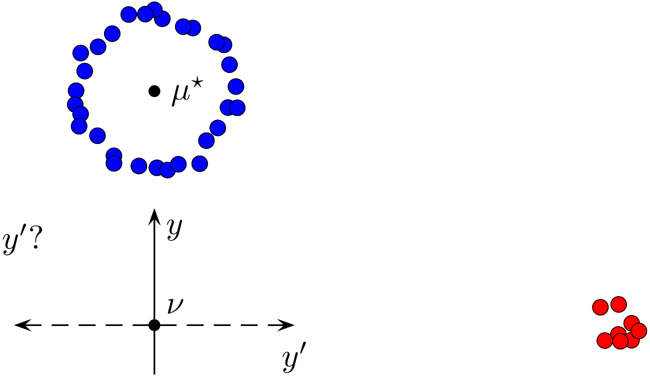
$$M = yy^\top$$

$$y \approx (\nu - \mu^\star)$$

Why is y better than y'?



How Far is $\mu^\star$: Optimal Value of the SDPs

Lemma

When $\|\nu - \mu^\star\|_2 \geq \ldots$,

$$1 + 0.99 \|\nu - \mu^\star\|_2^2 \leq OPT_\nu \leq 1 + 1.01 \|\nu - \mu^\star\|_2^2 .$$

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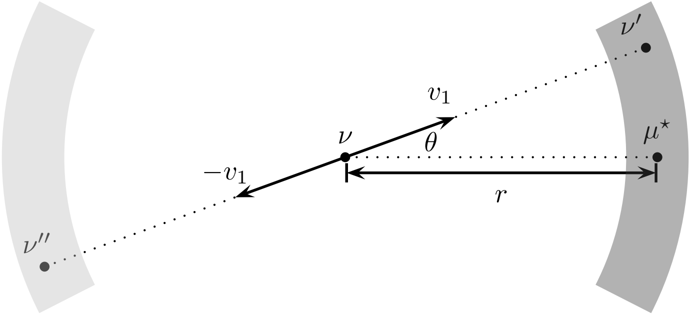

Full Algorithm: Sub-Gaussians

Algorithm 1: Robust Mean Estimation for Known Covariance Sub-Gaussian

Let ν be the coordinate-wise median of $\{X_i\}_{i=1}^N$;

for $i = 1$ **to** $O(\log d)$ **do**

Compute either

(i) a good solution $w \in \mathbb{R}^N$ for the primal SDP with parameters $(\nu, 2\epsilon)$; or

(ii) a good solution $M \in \mathbb{R}^{d \times d}$ for the dual SDP with parameters (ν, ϵ);

if *the objective value of w in primal SDP* $\leq 1 + c_0 \epsilon \ln(1/\epsilon)$ **then**

return *the weighted empirical mean* $\widehat{\mu}_w = \sum_{i=1}^N w_i X_i$;

else

Move ν closer to $\mu^\star$ using the top eigenvector of M.

Full Algorithm: Bounded Covariance

Algorithm 2: Robust Mean Estimation for Bounded Covariance Distributions

Let ν be the coordinate-wise median of $\{X_i\}_{i=1}^N$;

for $i = 1$ **to** $O(\log d)$ **do**

Compute either

(i) a good solution $w \in \mathbb{R}^N$ for the primal SDP with parameters $(\nu, 2\epsilon)$; or

(ii) a good solution $M \in \mathbb{R}^{d \times d}$ for the dual SDP with parameters (ν, ϵ);

if *the objective value of* w *in primal SDP is at most* c_1 **then**

return *the weighted empirical mean* $\widehat{\mu}_w = \sum_{i=1}^N w_i X_i$;

else

Move ν closer to $\mu^\star$ using the top eigenvector of M .

Summary: Robust Mean Estimation

Distribution	Error (δ)	# of Samples (N)	Runtime
Sub-Gaussian	$O(\epsilon\sqrt{\log(1/\epsilon)})$	$O(d/\delta^2)$	$\widetilde{O}(Nd/\epsilon^6)$
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We hope our work will serve as a starting point for the design of faster algorithms for high-dimensional robust estimation.

Follow up: Robust Covariance Estimation [C Diakonikolas Ge Woodruff '19]

Input: ϵ-corrupted set of N samples drawn from $\mathcal{N}(0, \Sigma)$.

Goal: Estimate Σ.

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All previous algorithms with similar error guarantee run in time $\Omega(d^{2\omega}) = \Omega(d^{4.74})$.

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Fast rectangular multiplication: $d \times d^2 \times d$ matrix multiplication can be done in time $O(d^{3.26})$.

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Computing the empirical covariance matrix $\frac{1}{N}\sum_{i=1}^{N} X_i X_i^\top$ takes $O(d^{3.26}/\epsilon^2)$ time.

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We use the primal-dual framework presented in this talk.

Naive implementation takes $\Omega(Nd^2) = \Omega(d^4)$ time. We need to open up the positive SDP solvers.

Open Problems

- Faster algorithms for other high-dimensional robust learning problems (e.g., sparse mean estimation / sparse PCA)?

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- Can we avoid the $\mathrm{poly}(1/\epsilon)$ in the runtime?