Computational-Statistical Tradeoffs and Open Problems

Ilias Diakonikolas (USC)

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**The Statistical Learning Problem**

- **Input:** sample generated by a **probabilistic model** with unknown $\theta^*$
- **Goal:** estimate parameters $\theta$ so that $\theta \approx \theta^*$

**Question 1:** Is there an **efficient** learning algorithm?

**Question 2:** Are there **tradeoffs** between these criteria?

**Main performance criteria:**
- Sample size
- Running time
- Robustness
OUTLINE

Part I: Computational Limits to Robust Estimation
  • Statistical Query Learning Model
  • Our Results
  • Generic Lower Bound Technique
  • Applications: Robust Mean Estimation & Learning GMMs

  • Part II: Future Directions
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STATISTICAL QUERIES [KEARNS’93]

\[ x_1, x_2, \ldots, x_m \sim D \text{ over } X \]
STATISTICAL QUERIES [KEARNS’93]

Problem $P \in \text{SQCompl}(q,m)$:
If exists a SQ algorithm that solves $P$ using $q$ queries to $\text{STAT}_D(\tau = 1/\sqrt{m})$

\[ \phi_1 : X \rightarrow [-1,1] \quad |v_1 - \mathbf{E}_{x \sim D}[\phi_1(x)]| \leq \tau \]
\[ \tau \text{ is tolerance of the query; } \tau = 1/\sqrt{m} \]
**Power of SQ Learning Algorithms**

- **Restricted Model**: Hope to prove unconditional computational lower bounds.

- **Powerful Model**: Wide range of algorithmic techniques in ML are implementable using SQs:
  - PAC Learning: $\text{AC}^0$, decision trees, linear separators, boosting.
  - Unsupervised Learning: stochastic convex optimization, moment-based methods, $k$-means clustering, EM, …
    ([Feldman-Grigorescu-Reyzin-Vempala-Xiao/JACM’17])
  - **Only known exception**: Gaussian elimination over finite fields (e.g., learning parities).

- For all problems in this talk, strongest known algorithms are SQ.
Methodology for Proving SQ Lower Bounds

Statistical Query Dimension:

- Fixed-distribution PAC Learning
  [Blum-Furst-Jackson-Kearns-Mansour-Rudich’95; ...]

- General Statistical Problems
  [Feldman-Grigorescu-Reyzin-Vempala-Xiao’13, ..., Feldman’16]

- Pairwise correlation between $D_1$ and $D_2$ with respect to $D$:

  $$\chi_D(D_1, D_2) := \int_{\mathbb{R}^d} D_1(x)D_2(x)/D(x)dx - 1$$

- **Fact**: Suffices to construct a large set of distributions that are *nearly* uncorrelated.
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**Generic SQ Lower Bound Construction**

General Technique for SQ Lower Bounds:
Leads to Tight Lower Bounds
for a range of High-dimensional Estimation Tasks

Concrete Applications of our Technique:

- Robustly Learning Mean and Covariance
- Learning Gaussian Mixture Models (GMMs)
- Statistical-Computational Tradeoffs (e.g., sparsity)
- Robustly Testing a Gaussian
Theorem: Suppose $d \geq \text{polylog}(1/\epsilon)$. Any SQ algorithm that learns an $\epsilon$-corrupted Gaussian $\mathcal{N}(\mu, I)$ in the strong contamination model within error $O(\epsilon \sqrt{\log(1/\epsilon)/M})$ requires either:

- SQ queries of accuracy $d^{-M/6}$

or

- At least $d^{\Omega(M^{1/2})}$ many SQ queries.

Take-away: Any asymptotic improvement in error guarantee over prior work requires super-polynomial time.
Theorem: Suppose that $d \geq \text{poly}(k)$. Any SQ algorithm that learns separated $k$-GMMs over $\mathbb{R}^d$ to constant error requires either:

- SQ queries of accuracy $d^{-k/6}$

or

- At least $2^{\Omega(d^{1/8})} \geq d^{2k}$ many SQ queries.

Take-away: Computational complexity of learning GMMs is inherently exponential in number of components.
### Applications: Concrete SQ Lower Bounds

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<td>Error: $O(\epsilon \log^{1/2}(1/\epsilon))$ [DKKLMS'16]</td>
<td>Runtime Lower Bound: $d^{\text{poly}(M)}$ for factor $M$ improvement in error.</td>
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<tr>
<td>Robust Gaussian Covariance Estimation</td>
<td>Error: $O(\epsilon \log(1/\epsilon))$ [DKKLMS'16]</td>
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<td>Learning $k$-GMMs (without noise)</td>
<td>Runtime: $d^{g(k)}$ [MV'10, BS'10]</td>
<td>Runtime Lower Bound: $d^{\Omega(k)}$</td>
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**General Recipe For SQ Lower Bounds**

- **Step #1:** Construct distribution $P_v$ that is standard Gaussian in all directions except $v$.

- **Step #2:** Construct the univariate projection in the $v$ direction so that it matches the first $m$ moments of $\mathcal{N}(0, 1)$.

- **Step #3:** Consider the family of instances $\mathcal{D} = \{P_v\}_v$

**Non-Gaussian Component Analysis** [Blanchard et al. 2006]
**Hidden Direction Distribution**

**Definition:** For a unit vector $v$ and a univariate distribution with density $A$, consider the high-dimensional distribution

$$P_v(x) = A(v \cdot x) \exp \left( -\|x - (v \cdot x)v\|^2 / 2 \right) / (2\pi)^{d-1}/2.$$

**Example:**

![Graphical representation of the hidden direction distribution](image)
**Definition:** For a unit vector $v$ and a univariate distribution with density $A$, consider the high-dimensional distribution

$$P_v(x) = A(v \cdot x) \exp \left(-\|x - (v \cdot x)v\|_2^2/2\right) / (2\pi)^{(d-1)/2}.$$ 

**Proposition:** Suppose that:
- $A$ matches the first $m$ moments of $\mathcal{N}(0,1)$
- We have $d_{TV}(P_v, P_{v'}) > 2\delta$ as long as $v, v'$ are nearly orthogonal.

Then any SQ algorithm that learns an unknown $P_v$ within error $\delta$ requires either queries of accuracy $d^{-m}$ or many queries.
WHY IS FINDING A HIDDEN DIRECTION HARD

Observation: Low-Degree Moments do not help.

• $A$ matches the first $m$ moments of $\mathcal{N}(0, 1)$
• The first $m$ moments of $P_v$ are identical to those of $\mathcal{N}(0, I)$
• Degree-$(m+1)$ moment tensor has $\Omega(d^m)$ entries.

Claim: Random projections do not help.

• To distinguish between $P_v$ and $\mathcal{N}(0, I)$, would need exponentially many random projections.
1-D Projections Are Almost Standard Gaussians

**Key Lemma:** Let $Q$ be the distribution of $v' \cdot X$, where $X \sim P_v$. Then, we have that:

$$\chi^2(Q, \mathcal{N}(0, 1)) \leq (v \cdot v')^{2(m+1)} \chi^2(A, \mathcal{N}(0, 1))$$

- $X \sim P_v$
- $v' \cdot X \sim Q$
- $v \cdot X \sim A$
- $v^\perp \cdot X \sim \mathcal{N}(0, 1)$
Proof of Key Lemma (I)

\[ Q(x') = \int_{\mathbb{R}} A(x) G(y) dy' \]

\[ X \sim P_v \]
\[ v' \cdot X \sim Q \]
\[ v \cdot X \sim A \]
\[ v^\perp \cdot X \sim \mathcal{N}(0,1) \]
Proof of Key Lemma (I)

\[ Q(x') = \int_{\mathbb{R}} A(x)G(y)dy' \]

\[ = \int_{\mathbb{R}} A(x' \cos \theta + y' \sin \theta)G(x' \sin \theta - y' \cos \theta)dy' \]

\[ X \sim P_v \]

\[ v' \cdot X \sim Q \]

\[ v \cdot X \sim A \]

\[ v^\perp \cdot X \sim \mathcal{N}(0, 1) \]
**Proof of Key Lemma (II)**

\[ Q(x') = \int_{\mathbb{R}} A(x' \cos \theta + y' \sin \theta) G(x' \sin \theta - y' \cos \theta) dy' \]

\[ = (U_\theta A)(x') \]

where \( U_\theta \) is the operator over \( f : \mathbb{R} \to \mathbb{R} \)

\[ U_\theta f(x) := \int_{y \in \mathbb{R}} f(x \cos \theta + y \sin \theta) G(x \sin \theta - y \cos \theta) dy \]

**Gaussian Noise (Ornstein-Uhlenbeck) Operator**
EIGEN-DECOMPOSITION OF ORNSTEIN-UHLENBECK OPERATOR

Linear Operator $U_\theta$ acting on functions $f : \mathbb{R} \to \mathbb{R}$

$$U_\theta f(x) := \int_{y \in \mathbb{R}} f(x \cos \theta + y \sin \theta)G(x \sin \theta - y \cos \theta)dy$$

Fact (Mehler'66): $U_\theta (He_i G)(x) = \cos^i(\theta)He_i(x)G(x)$

- $He_i(x)$ denotes the degree-$i$ Hermite polynomial.
- Note that $\{He_i(x)G(x)/\sqrt{i!}\}_{i \geq 0}$ are orthonormal with respect to the inner product
  $$\langle f, g \rangle = \int_{\mathbb{R}} f(x)g(x)/G(x)dx$$
**Generic SQ Lower Bound**

**Definition:** For a unit vector \( v \) and a univariate distribution with density \( A \), consider the high-dimensional distribution

\[
P_v(x) = A(v \cdot x) \exp \left( -\|x - (v \cdot x)v\|_2^2/2 \right) / (2\pi)^{(d-1)/2}.
\]

**Proposition:** Suppose that:
- \( A \) matches the first \( m \) moments of \( \mathcal{N}(0, 1) \)
- We have \( d_{TV}(P_v, P_{v'}) > 2\delta \) as long as \( v, v' \) are nearly orthogonal.

Then any SQ algorithm that learns an unknown \( P_v \) within \( \delta \) error requires either queries of accuracy \( d^{-m} \) or \( 2^{d^{\Omega(1)}} \) many queries.
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SQ LOWER BOUND FOR ROBUST MEAN ESTIMATION (I)

Want to show:

Theorem: Any SQ algorithm that learns an $\epsilon$-corrupted Gaussian in the strong contamination model within error $\epsilon \sqrt{\log(1/\epsilon)/M}$ requires either SQ queries of accuracy $d^{-M/6}$ or at least $d^{\Omega(M^{1/2})}$ many SQ queries.

by using our generic proposition:

Proposition: Suppose that:
- $A$ matches the first $m$ moments of $\mathcal{N}(0, 1)$
- We have $d_{TV}(P_v, P_{v'}) > 2\delta$ as long as $v, v'$ are nearly orthogonal.

Then any SQ algorithm that learns an unknown $P_v$ within error $\delta$ requires either queries of accuracy $d^{-m}$ or $2d^{\Omega(1)}$ many queries.
**SQ LOWER BOUND FOR ROBUST MEAN ESTIMATION (II)**

**Proposition:** Suppose that:
- $A$ matches the first $m$ moments of $\mathcal{N}(0, 1)$
- We have $d_{TV}(P_v, P_{v'}) > 2\delta$ as long as $v, v'$ are nearly orthogonal.

Then any SQ algorithm that learns an unknown $P_v$ within error $\delta$ requires either queries of accuracy $d^{-m}$ or $2^d\Omega(1)$ many queries.

**Lemma:** There exists a univariate distribution $A$ that is $\epsilon$-close to $\mathcal{N}(\mu, 1)$ such that:
- $A$ agrees with $\mathcal{N}(0, 1)$ on the first $M$ moments.
- We have that $\mu = \Omega(\epsilon \sqrt{\log(1/\epsilon)}/M^2)$
- Whenever $v$ and $v'$ are nearly orthogonal $d_{TV}(P_v, P_{v'}) = \Omega(\mu)$.
**SQ Lower Bound for Learning GMMs (I)**

Want to show:

**Theorem:** Any SQ algorithm that learns separated $k$-GMMs over $\mathbb{R}^d$ to constant error requires either SQ queries of accuracy $d^{-k/6}$ or at least $2^{\Omega(d^{1/8})} \geq d^{2k}$ many SQ queries.

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**Proposition:** Suppose that:
- $A$ matches the first $m$ moments of $\mathcal{N}(0, 1)$
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**SQ Lower Bound for Learning GMMs (II)**

**Proposition:** Suppose that:
- $A$ matches the first $m$ moments of $\mathcal{N}(0, 1)$
- We have $d_{TV}(P_v, P_{v'}) > 2\delta$ as long as $v$, $v'$ are nearly orthogonal.

Then any SQ algorithm that learns an unknown $P_v$ within error $\delta$ requires either queries of accuracy $d^{-m}$ or $2^d\Omega(1)$ many queries.

**Lemma:** There exists a univariate distribution $A$ that is a $k$-GMM with components $A_i$ such that:
- $A$ agrees with $\mathcal{N}(0, 1)$ on the first $2k-1$ moments.
- Each pair of components are separated.
- Whenever $v$ and $v'$ are nearly orthogonal $d_{TV}(P_v, P_{v'}) \geq 1/2$.
Lemma: There exists a univariate distribution $A$ that is a $k$-GMM with components $A_i$ such that:

- $A$ agrees with $\mathcal{N}(0, 1)$ on the first $2k-1$ moments.
- Each pair of components are separated.
- Whenever $v$ and $v'$ are nearly orthogonal $d_{TV}(P_v, P_{v'}) \geq 1/2$. 

$$A$$
SQ LOWER BOUND FOR LEARNING GMMs (IV)

High-Dimensional Distributions $P_v$ look like “parallel pancakes”:

Efficiently learnable for $k=2$. [Brubaker-Vempala’08]
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Future Directions: Computational Lower Bounds

- General Technique to Prove SQ Lower Bounds
- Robustness can make high-dimensional estimation harder computationally and information-theoretically.

Future Directions:

- Further Applications of our Framework
  - List-Decodable Mean Estimation [D-Kane-Stewart’18]
  - Robust Regression [D-Kong-Stewart’18]
  - Adversarial Examples [Bubeck-Price- Razenshteyn’18]
  - Discrete Distributions [D-Gouleakis-Kane-Stewart’19]

- Alternative Evidence of Computational Hardness?
  - SoS Lower Bounds
  - Reductions from Average-Case Problems (e.g., Planted Clique, R-3SAT)
  - Reductions from Worst-case Problems? First step: [Hopkins-Li, COLT’19]
Future Directions: Algorithms

- Pick your favorite high-dimensional probabilistic model for which a (non-robust) efficient learning algorithm is known.
- Make it robust!
CONCRETE ALGORITHMIC OPEN PROBLEMS

Open Problem 1: Robustly Estimating Gaussian Covariance Within Error $O(\epsilon)$ in Additive Contamination Model (Huber’s Model)

Currently Best Known Algorithm [DKKLMS’18] runs in time $\text{poly}(d) \cdot (1/\epsilon)^{\text{polylog}(1/\epsilon)}$.

Open Problem 2: Robustly Learn a Mixture of 2 Arbitrary Gaussians

Spherical components: [Diakonikolas-Kane-Stewart’18, Hopkins-Li’18, Kothari-Steinhardt’18]
Filtering for robust mean estimation is practical, but runtime is super-linear $\tilde{\Theta}(N d^2)$.

**Question:** Can we design near-linear time algorithms?

- Robust Mean Estimation:
  - [Cheng-D-Ge, SODA’19] $\tilde{\Theta}(N d/\text{poly}(\epsilon))$.
  - [Depersin-Lecue, Arxiv-June 2019] $\tilde{\Theta}(N d)$.
  - [Dong-Hopkins-Li, upcoming] $\tilde{\Theta}(N d)$.

- How about more general estimation tasks?
  - [Cheng-D-Ge-Woodruff, COLT’19]
  - Robust Sparse Estimation?
  - List-Decodable Learning?
Broader Research Directions

How can we robustly learn rich representations of data, based on natural hypotheses about the structure in data?

Can we robustly test our hypotheses about structure in data before learning?

Broader Challenges:

- Richer Families of Problems and Models
- Connections to Non-convex Optimization, Adversarial Examples, GANs, …
- Relation to Related Notions of Algorithmic Stability
  (Differential Privacy, Adaptive Data Analysis)
- Further Applications (ML Security, Computer Vision, …)
- Other notions of robustness?

Thank you!
Questions?