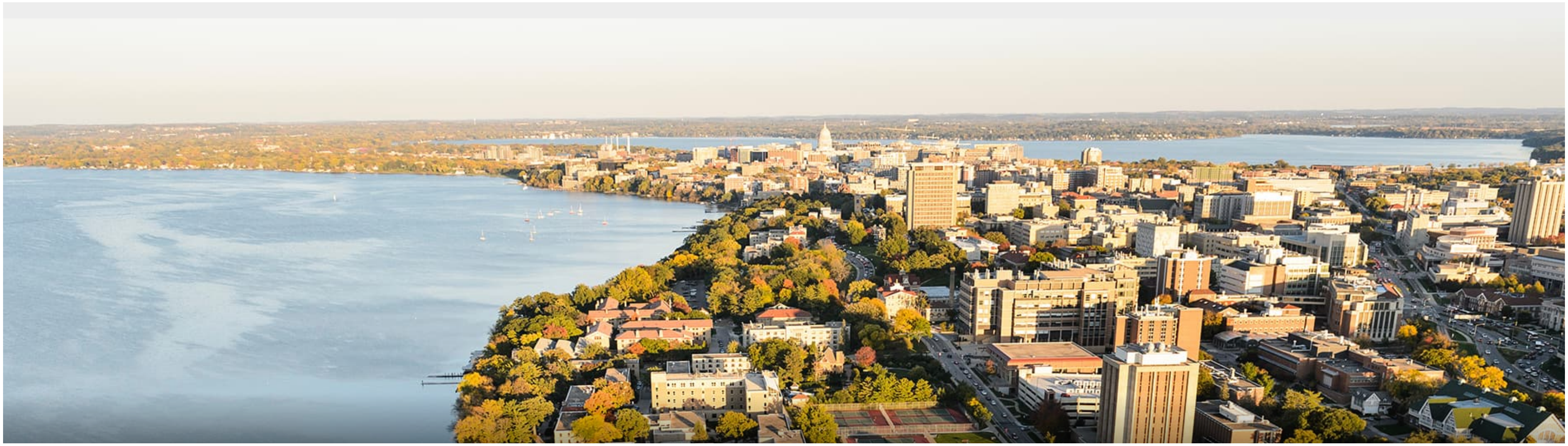

CS 760: Machine Learning
Probability & Graphical Models

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Nov. 8, 2022

Announcements

- **Logistics**:
  - HW 5 released later this week

- Class roadmap:

Thursday, Nov. 10	Graphical Models I
Tuesday	Graphical Models II
Thursday	Less-than-full Supervision
Tuesday	Unsupervised Learning I

Outline

- **Review, SVMs, Kernels**
  - Duality, feature maps, kernel trick
- **Probability Tutorial**
  - Basics, joint probability, conditional probabilities, etc
- **Bayesian Networks**
  - Definition, examples, inference

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Review: Constrained Optimization & Duality

$$\min_w \; f(w)$$ ← Objective

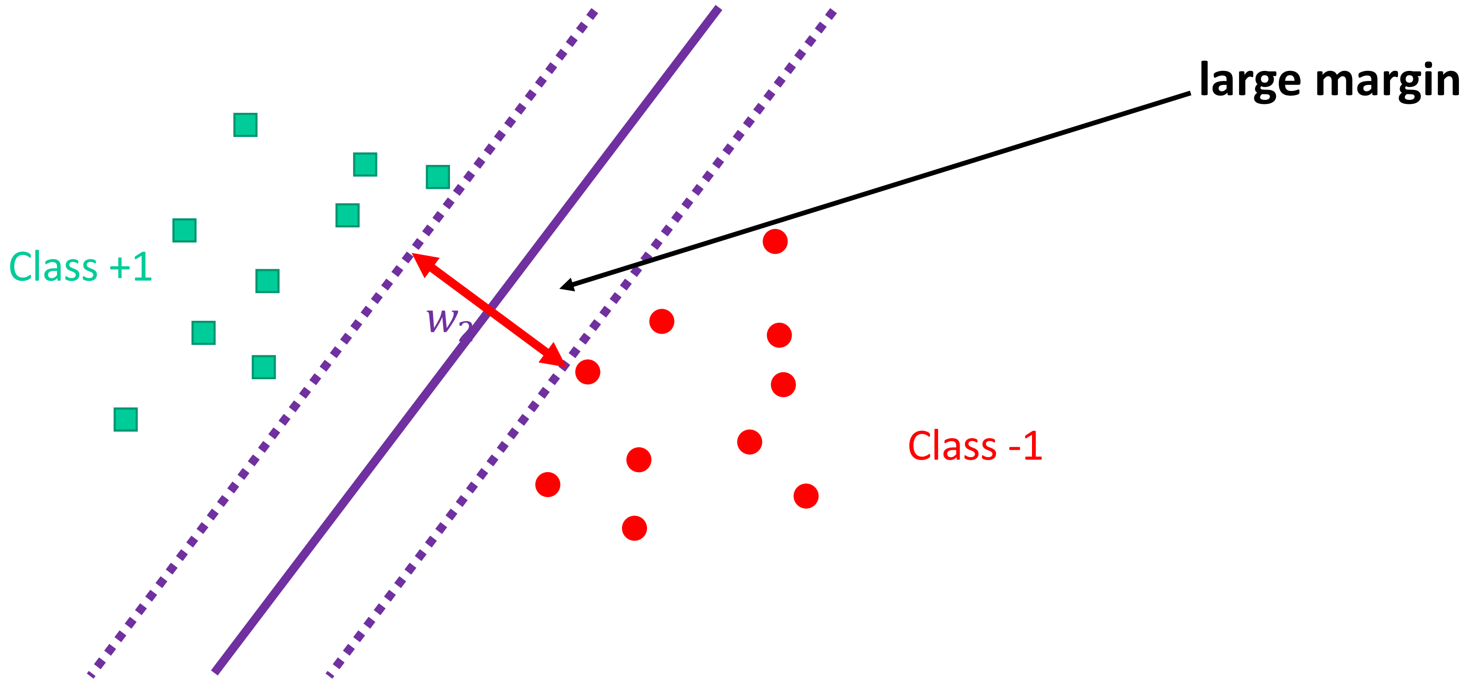
$$g_i(w) \leq 0, \forall 1 \leq i \leq k$$
$$h_j(w) = 0, \forall 1 \leq j \leq l$$ ← Constraints

- **Lagrangian**: $\mathcal{L}(w, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \; f(w) + \sum_i \alpha_i g_i(w) + \sum_j \beta_j h_j(w)$

- Primal problem $p^* \coloneqq \min_w f(w) = \min_w \max_{\boldsymbol{\alpha},\boldsymbol{\beta}:\alpha_i \geq 0} \mathcal{L}(w, \boldsymbol{\alpha}, \boldsymbol{\beta})$
- Dual problem $d^* \coloneqq \max_{\boldsymbol{\alpha},\boldsymbol{\beta}:\alpha_i \geq 0} \min_w \mathcal{L}(w, \boldsymbol{\alpha}, \boldsymbol{\beta})$
- Always true: $d^* \leq p^*$

Review: Apply to Training Linear Classifier

- Want: a **large margin**



Review: Support Vector Machines Goal

Define the margin to be

$$\gamma = \min_i \frac{y_i f_{w,b}(x_i)}{||w||}$$

← **We proved this**

- If $f_{w,b}$ incorrect on some x_i, the margin is **negative**
- Fix scale: $y_{i^*}(w^T x_{i^*} + b) = 1$. Then, margin overall is $\frac{1}{||w||}$

Primal problem:

$$\min_{w,b} \frac{1}{2} ||w||^2$$

← **Objective: Large margin**

$$y_i(w^T x_i + b) \geq 1, \forall i$$

← **Constraints: Correct on training data**

SVM: Dual Version

- Reduces to dual problem:

$$\max_{\boldsymbol{\alpha}} \mathcal{L}(w, b, \boldsymbol{\alpha}) = \max_{\boldsymbol{\alpha}} \sum_i \alpha_i - \frac{1}{2} \sum_{ij} \alpha_i \alpha_j y_i y_j x_i^T x_j$$

$$\sum_i \alpha_i y_i = 0, \alpha_i \geq 0$$

Note: only variables are primal

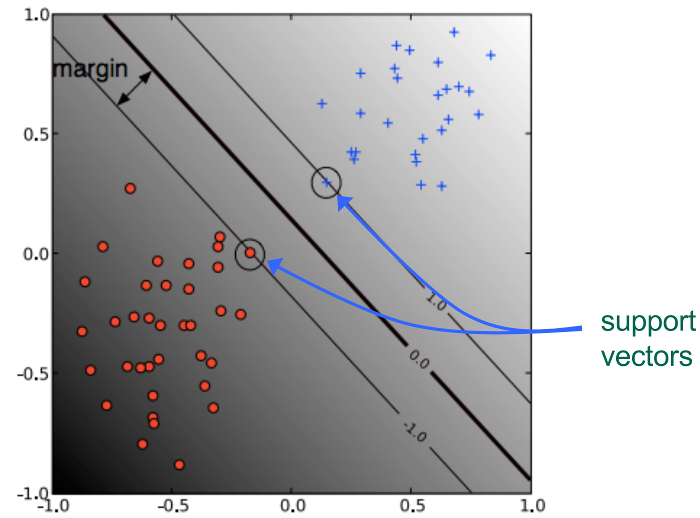
- Since $w = \sum_i \alpha_i y_i x_i$, we have $w^T x + b = \sum_i \alpha_i y_i x_i^T x + b$
- Note: only deals with data via **inner products** $x_i^T x_j$

SVM: Support Vectors

- Solution is a sparse linear combination of training instances

$$w = \sum_i \alpha_i y_i x_i$$

- Those instances with $\alpha_i > 0$ are called ***support vectors***
  - Lie on the margin boundary
- Solution does not change if we delete instances with $\alpha_i = 0$



SVM: Soft Margin

What if our data isn't linearly separable?

- Can adjust our approach by using *slack variables* (denoted by ζ_i) to tolerate errors

$$\min_{w,b,\zeta_i} \frac{1}{2} ||w||^2 + C \sum_i \zeta_i$$

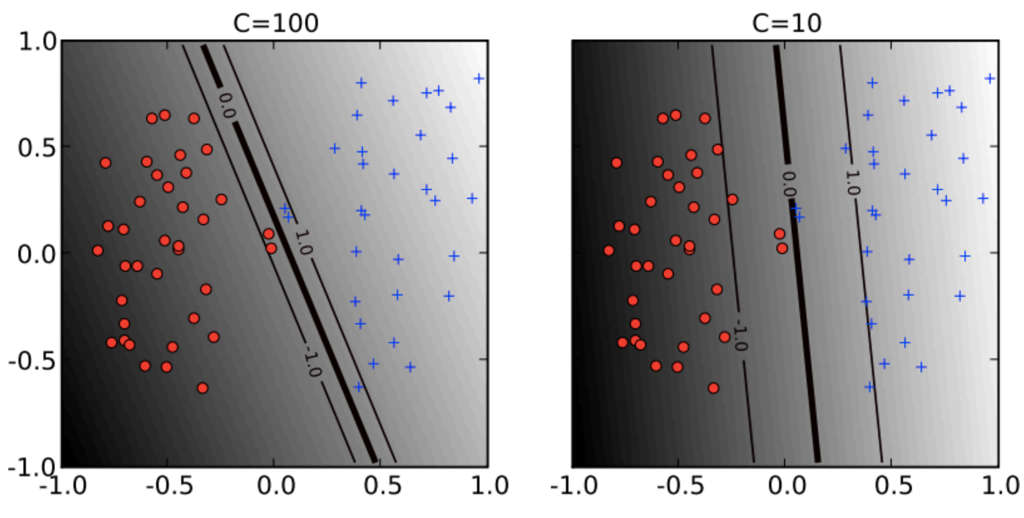
$$y_i(w^T x_i + b) \geq 1 - \zeta_i, \zeta_i \geq 0, \forall i$$

- C determines the relative importance of maximizing margin vs. minimizing slack

SVM: Soft Margin

$$\min_{w,b,\zeta_i} \frac{1}{2}||w||^2 + C\sum_i \zeta_i$$

$$y_i(w^T x_i + b) \geq 1 - \zeta_i, \zeta_i \geq 0, \forall i$$



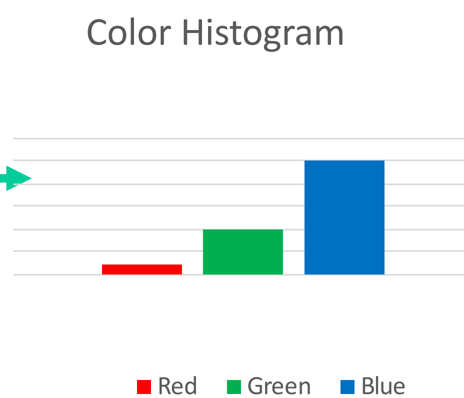
Ben-Hur & Weston, *Methods in Molecular Biology* 2010

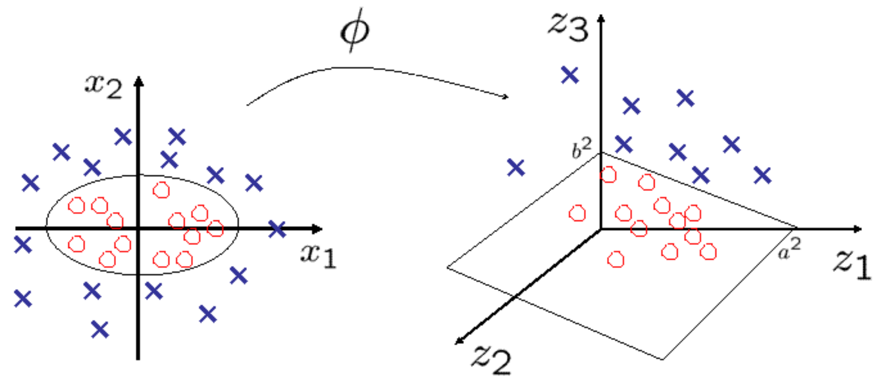
Feature Maps

- Can take a set of features and map them into another
  - Can also construct non-linear features
  - Use these inside a linear classifier?

x $\phi(x)$

Extract features





$$\phi : (x_1, x_2) \longrightarrow (x_1^2, \sqrt{2}x_1x_2, x_2^2)$$

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = 1 \longrightarrow \frac{z_1}{a^2} + \frac{z_3}{b^2} = 1$$

Feature Maps and SVMs

Want to use feature space $\{\phi(x_i)\}$ in linear classifier…

- Downside: dimension might be high (even infinite!)
- So we don't want to write down $\phi(x_i) = [0.2, 0.3, \dots]$

Recall our SVM dual form:

- Only relies on inner products $x_i^T x_j$

$$\mathcal{L}(w, b, \boldsymbol{\alpha}) = \sum_i \alpha_i - \frac{1}{2} \sum_{ij} \alpha_i \alpha_j y_i y_j x_i^T x_j$$

$$\sum_i \alpha_i y_i = 0, \alpha_i \geq 0$$

Kernel Trick

- Using SVM on the feature space $\{\phi(x_i)\}$: only need $\phi(x_i)^T\phi(x_j)$

- Conclusion: no need to design $\phi(\cdot)$, only need to design

$$k(x_i, x_j) = \phi(x_i)^T\phi(x_j)$$

Kernel Matrix Feature Maps

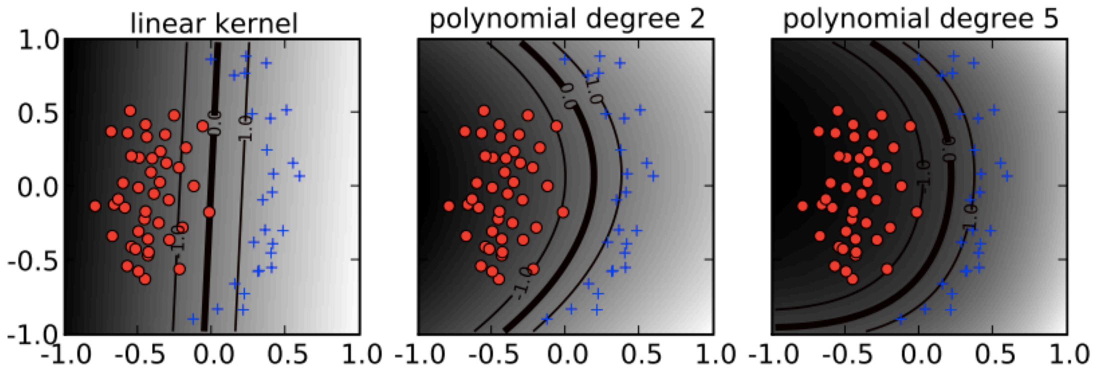
Kernel Types: Polynomial

- Fix degree d and constant c:

$$k(x, x') = (x^T x' + c)^d$$

- What are $\phi(x)$?
- Expand the expression to get $\phi(x)$

$$\forall \mathbf{x}, \mathbf{x}' \in \mathbb{R}^2, \quad K(\mathbf{x}, \mathbf{x}') = (x_1 x_1' + x_2 x_2' + c)^2 = \begin{bmatrix} x_1^2 \\ x_2^2 \\ \sqrt{2}\, x_1 x_2 \\ \sqrt{2c}\, x_1 \\ \sqrt{2c}\, x_2 \\ c \end{bmatrix} \cdot \begin{bmatrix} {x'}_1^2 \\ {x'}_2^2 \\ \sqrt{2}\, x_1' x_2' \\ \sqrt{2c}\, x_1' \\ \sqrt{2c}\, x_2' \\ c \end{bmatrix}$$



Ben-Hur & Weston, *Methods in Molecular Biology* 2010

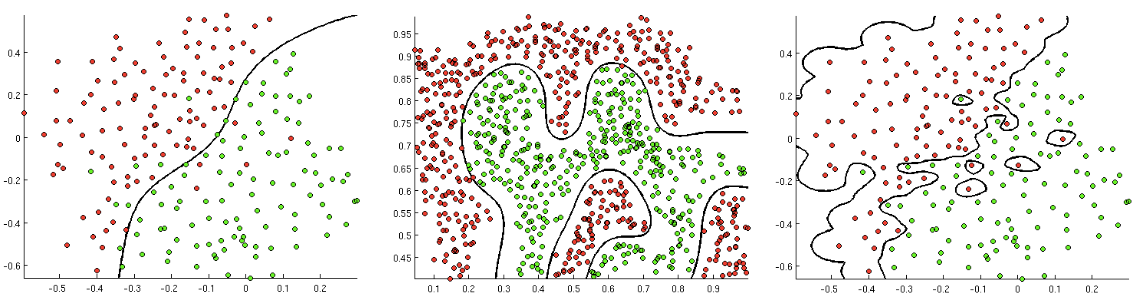
Kernel Types: Gaussian/RBF

- Fix bandwidth σ:

$$k(x, x') = \exp(-||x - x'||^2 / 2\sigma^2)$$

  - Also called radial basis function (RBF) kernels

$\gamma = 10$ $\quad\quad$ $\gamma = 100$ $\quad\quad$ $\gamma = 1000$



$$k(x, x') = \exp(-\gamma||x - x'||^2)$$

Andrew Ng

Theory of Kernels

- Part of a deep mathematical theory
- With some conditions, any kernel yields a feature map:
  - Theorem: $k(x, x')$ has expansion

$$k(x, x') = \sum_{i}^{+\infty} a_i \phi_i(x) \phi_i(x')$$

← **Feature Maps**

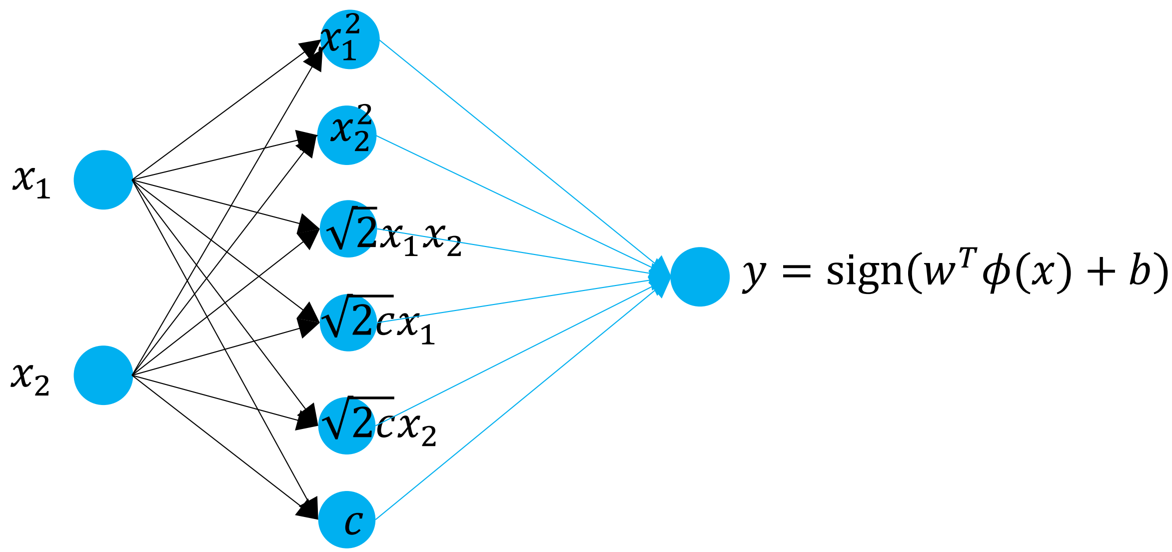
for nonnegative a_i's, if and only if for any function $c(x)$,

$$\int \int c(x) c(x') k(x, x') dx dx' \geq 0$$

- Given certain requirements/conditions, can construct a bunch of new kernels from existing ones

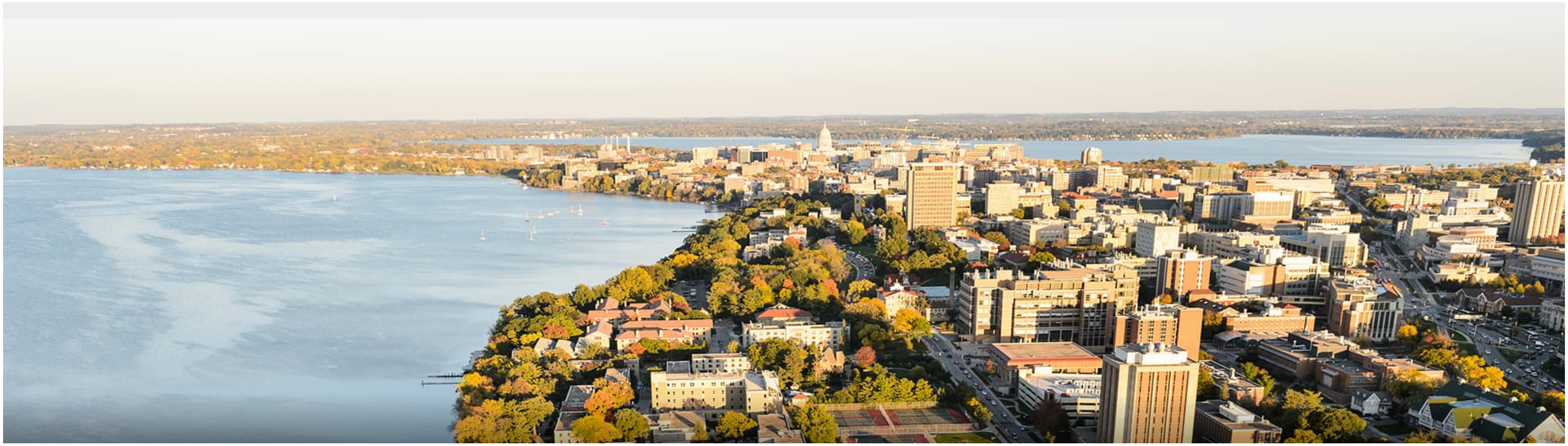
Kernel Methods VS Neural Networks

- Can think of our kernel SVM approach as fixing a layer of a neural network



SVM Review

- Can find globally optimal solutions: convex optimization
  - No local minima (unlike training general NNs)
- Can train primal or dual
  - Dual: relies on **support vectors**; enables use of **kernels**
- Variety of pre-existing optimization techniques
- Kernels: allow non-linear decision boundaries
  - And to represent all sorts of new data (strings, trees)
  - High-dimensional representations, but can use kernel trick to avoid explicitly computing feature maps
  - Good performance! Sometimes close to DNNs



Break & Quiz

Q1-1: Are these statements true or false?
(A) If we have multiple optimal solutions on a given training set, those solutions will also have the same test loss.
(B) If a hyperplane only changes its bias term by 1, then the distance from some point x to the hyperplane will not change.

1. True, True
2. True, False
3. False, True
4. False, False

Q1-1: Are these statements true or false?
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1. True, True
2. True, False
3. False, True
4. False, False ⟵

(A) Multiple optimal solutions on the training usually have different test loss. Please refer to the example given in the lecture.
(B) Recall that the distance is given by $\frac{|f_{w,b}(x)|}{\|w\|}$. If only the bias term is changed, then $|f_{w,b}(x)|$ will change while $\| w \|$ remains same. So the distance will also be changed.

Q1-2: Are these statements true or false?

(A) Define the margin to be $\gamma = \min_i \frac{y_i f_{w,b}(x_i)}{\|w\|}$, if $f_{w,b}(\mathrm{x})$ predicts correctly on some x_i and incorrectly on others, then the margin will be positive.

(B) If the training set can be correctly separated, then $\max_{w,b} \gamma$ can still be negative.

1. True, True
2. True, False
3. False, True
4. False, False

Q1-2: Are these statements true or false?

(A) Define the margin to be $\gamma = \min_i \frac{y_i f_{w,b}(x_i)}{\|w\|}$, if $f_{w,b}(\mathrm{x})$ predicts correctly on some x_i and incorrectly on others, then the margin will be positive.

(B) If the training set can be correctly separated, then $\max_{w,b} \gamma$ can still be negative.

1. True, True
2. True, False
3. False, True
4. False, False

(A) In this case, $\frac{y_i f_{w,b}(x_i)}{\|w\|}$ would be negative on those x_i with incorrect predictions. So take min on all training data, we will get the margin negative.

(B) In this case, there exists at least one w and b such that all instances are correctly classified, so the corresponding margin is non-negative.

Q1-3: Are these statements true or false?
(A) The solution of SVM will always change if we remove some instances from the training set.
(B) If we can only access the labels and the inner products of instances $\{x_i{}^T x_j\}_{i,j}$, we can NOT solve the learning problem in SVM.

1. True, True
2. True, False
3. False, True
4. False, False

Q1-3: Are these statements true or false?
(A) The solution of SVM will always change if we remove some instances from the training set.
(B) If we can only access the labels and the inner products of instances $\{x_i^T x_j\}_{i,j}$, we can NOT solve the learning problem in SVM.

1. True, True
2. True, False
3. False, True
4. False, False ←

(A) As is shown in the lecture, if we remove those instances with $\alpha_i = 0$, it will not influence the SVM result.
(B) We can see that the dual problem only depends on y_i and the inner products of training instances. So we can also solve the SVM problem in this case.

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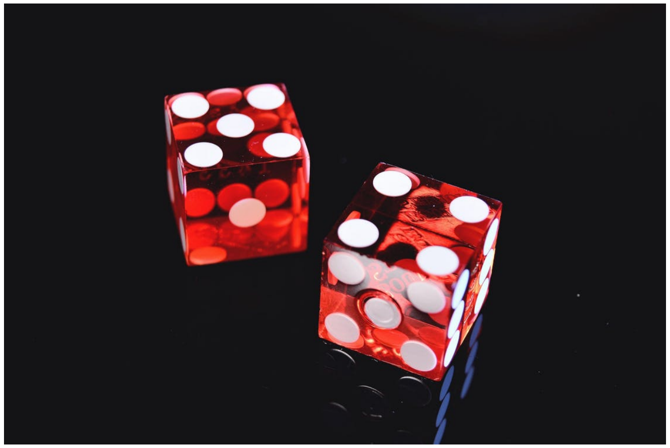
Probability Tutorial: Outcomes & Events

- Outcomes: possible results of an **experiment**
- **Events:** subsets of outcomes we're interested in

Ex:

$$\Omega = \underbrace{\{1, 2, 3, 4, 5, 6\}}_{\text{outcomes}}$$

$$\mathcal{F} = \underbrace{\{\emptyset, \{1\}, \{2\}, \ldots, \{1, 2\}, \ldots, \Omega\}}_{\text{events}}$$



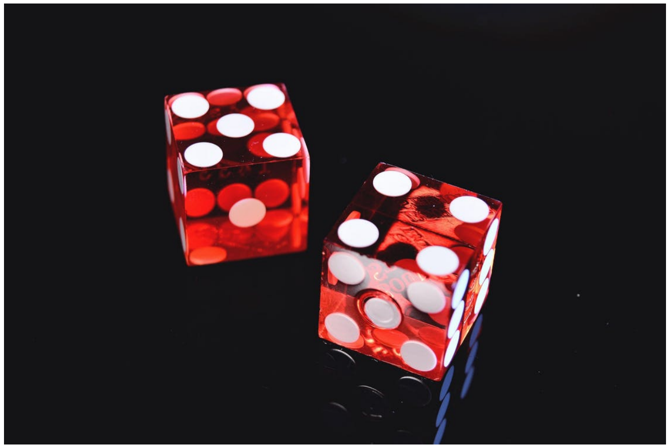
Probability Tutorial: Outcomes & Events

- Event space can be smaller:

$$\mathcal{F} = \underbrace{\{\emptyset, \{1,3,5\}, \{2,4,6\}, \Omega\}}_{\text{events}}$$

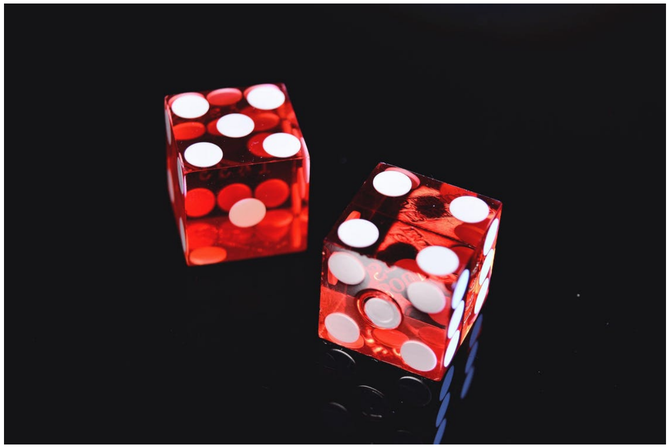
- Two components always in it!

$$\emptyset, \Omega$$



Probability Tutorial: Sigma Fields

- F is a "sigma algebra".
  - Follows certain rules:
    - Everything in it (saw this already)
    - If A is in F, so is A^c
    - Closed under countable unions



Probability Tutorial: Probability Spaces

- Now we need a way to produce probabilities of events, so introduce a function

$$P : \mathcal{F} \to [0, 1]$$

- Has certain properties, which we'll see in a second.
- Overall, we get a probability space

$$(\Omega, \mathcal{F}, P)$$

Probability Tutorial: Probability Spaces

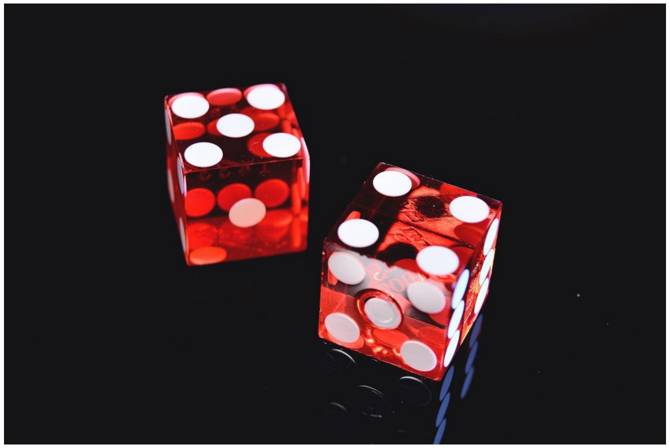
- We have outcomes and events and probabilities
- I.e.,

$$\text{For } E \in \mathcal{F},\ P(E) \in [0,1]$$

Back to our example:

$$\mathcal{F} = \underbrace{\{\emptyset, \{1,3,5\}, \{2,4,6\}, \Omega\}}_{\text{events}}$$

$$P(\{1,3,5\}) = 0.2, P(\{2,4,6\}) = 0.8$$



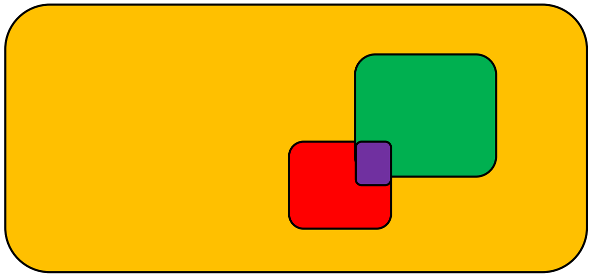
Basics: **Axioms**

- Rules for probability:
  - For all events $E \in \mathcal{F}, P(E) \geq 0$
  - Always, $P(\emptyset) = 0, P(\Omega) = 1$
  - For disjoint events,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

- Easy to derive other laws. Ex: non-disjoint events

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

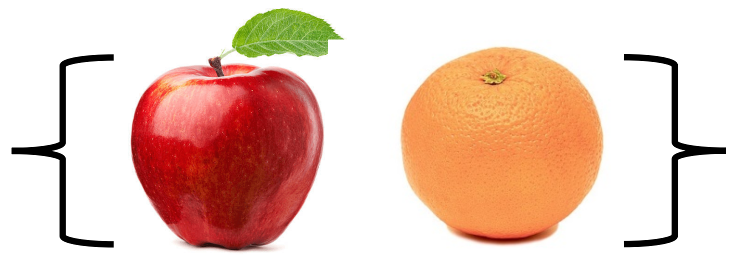


Basics: **Random Variables**

- Really, functions
- Map outcomes to real values

$$X : \Omega \to \mathbb{R}$$

- Why?
  - So far, everything is a set.
  - Hard to work with!
  - Real values are easy to work with



- One requirement, "F measurable". For any *c*,

$$\{\omega : X(\omega) \leq c\} \in \mathcal{F}$$

Basics: **CDF & PDF**

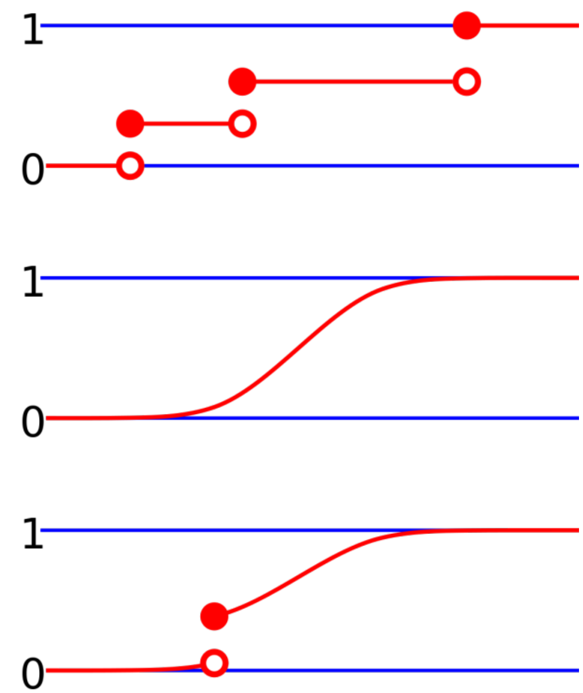
- Can still work with probabilities:

$$P(X = 3) := P(\{\omega : X(\omega) = 3\})$$

- Cumulative Distribution Func. (CDF)

$$F_X(x) := P(X \leq x)$$

- Density / mass function $p_X(x)$
  - Doesn't always exist!



Wiki CDF

Basics: **Expectation** & **Variance**

- Another advantage of RVs are ``summaries''
- Expectation:
  - The "average" $E[X] = \sum_a a \times P(x = a)$
- Variance: $Var[X] = E[(X - E[X])^2]$
  - A measure of spread
- Raw moments: $E[X], E[X^2], E[X^3], \ldots$
- Note: also don't always exist…
  - **Ex**: Cauchy distribution

Basics: **Expectation** Properties

- Expectation has very useful properties…
  - Linearity: $$E[\sum_i a_i X_i] = \sum_i a_i E[X_i]$$
  - Independence not required!
  - Hat check problem:
    - There is a dinner party where n people check their hats. The hats are mixed up during dinner, so that afterward each man receives a random hat. In particular, each person gets their own hat with probability 1/n. What is the expected number of people who get their own hat?

Basics: **Joint Distributions**

- Move from one variable to several
- Joint distribution

$$P(X = a, Y = b)$$

- Or more variables.

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k)$$

Basics: **Marginal** Probability

- Given a joint distribution

$$P(X = a, Y = b)$$

  - Get the distribution in just one variable:

$$P(X = a) = \sum_b P(X = a, Y = b)$$

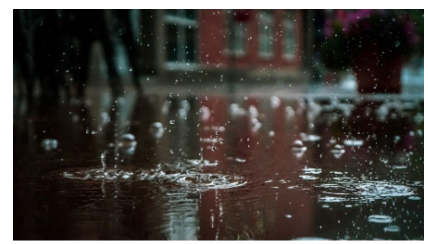
  - This is the “marginal” distribution.

Basics: **Marginal** Probability

$$P(X = a) = \sum_b P(X = a, Y = b)$$

	Sunny	Cloudy	Rainy
hot	150/365	40/365	5/365
cold	50/365	60/365	60/365

$$[P(\text{hot}), P(\text{cold})] = [\frac{195}{365}, \frac{170}{365}]$$



Independence

- Independence for a set of events $A_1, \ldots, A_k$

$$P(A_{i_1} A_{i_2} \cdots A_{i_j}) = P(A_{i_1}) P(A_{i_2}) \cdots P(A_{i_j})$$

for all the $i_1, \ldots, i_j$ combinations

- Why useful? Dramatically reduces the complexity
- Collapses joint into **product** of marginals
  - Note sometimes we have only pair-wise, etc independence

Uncorrelatedness

- For random variables, uncorrelated means

$$E[XY] = E[X]E[Y]$$

Note: weaker than independence.

- Independence implies uncorrelated (easy to see)
- Other way around: usually false (but not always).
- If X,Y independent, functions are not correlated:

$$E[f(X)f(Y)] = E[f(X)]E[f(Y)]$$

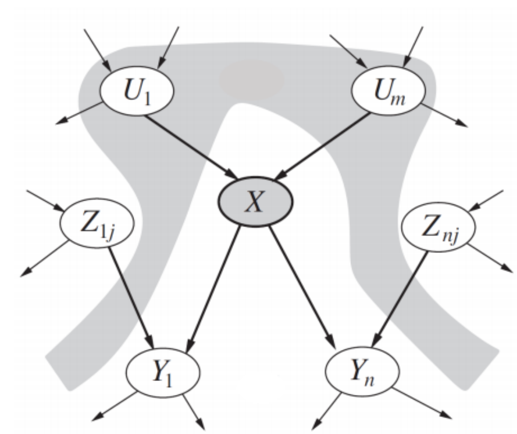
Conditional Probability

- For when we know something,

$$P(X = a|Y = b) = \frac{P(X = a, Y = b)}{P(Y = b)}$$

- Leads to **conditional independence**

$$P(X, Y|Z) = P(X|Z)P(Y|Z)$$



Credit: Devin Soni

Chain Rule

- Apply repeatedly,

$$P(A_1, A_2, \ldots, A_n)$$
$$= P(A_1)P(A_2|A_1)P(A_3|A_2, A_1) \ldots P(A_n|A_{n-1}, \ldots, A_1)$$

- Note: still big!
  - If some **conditional independence**, can factor!
  - Leads to **probabilistic graphical models (this lecture)**

Law of Total Probability

- Partition the sample space into disjoint B_1, …, B_k
- Then,

$$P(A) = \sum_i P(A|B_i)P(B_i)$$

- Useful way to control A via conditional probabilities.
  - **Example**: there are 5 red and 2 green balls in an urn. A random ball is selected and replaced by a ball of the other color; then a second ball is drawn. What is the probability the second ball is red?

Bayesian Inference

- Conditional Prob. & Bayes:

$$P(H|E_1, E_2, \ldots, E_n) = \frac{P(E_1, \ldots, E_n|H)P(H)}{P(E_1, E_2, \ldots, E_n)}$$

- Has more evidence.
  - Likelihood is hard---but **conditional independence assumption**

$$P(H|E_1, E_2, \ldots, E_n) = \frac{P(E_1|H)P(E_2|H)\cdots, P(E_n|H)P(H)}{P(E_1, E_2, \ldots, E_n)}$$

Random Vectors & Covariance

- Recall variance: $\mathbb{E}[(X - E[X])^2]$
- Now, for a **random vector** (same as joint of *d* RVs)
  - Note: size *d x d.* All variables are centered

$$\Sigma = \begin{bmatrix} \mathbb{E}[(X_1 - \mathbb{E}[X_1])^2] & \dots & [(X_1 - \mathbb{E}[X_1])((X_n - \mathbb{E}[X_n])] \\ \vdots & \vdots & \vdots \\ [(X_n - \mathbb{E}[X_n])((X_1 - \mathbb{E}[X_1])] & \dots & \mathbb{E}[(X_n - \mathbb{E}[X_n])^2] \end{bmatrix}$$

Cross-variance

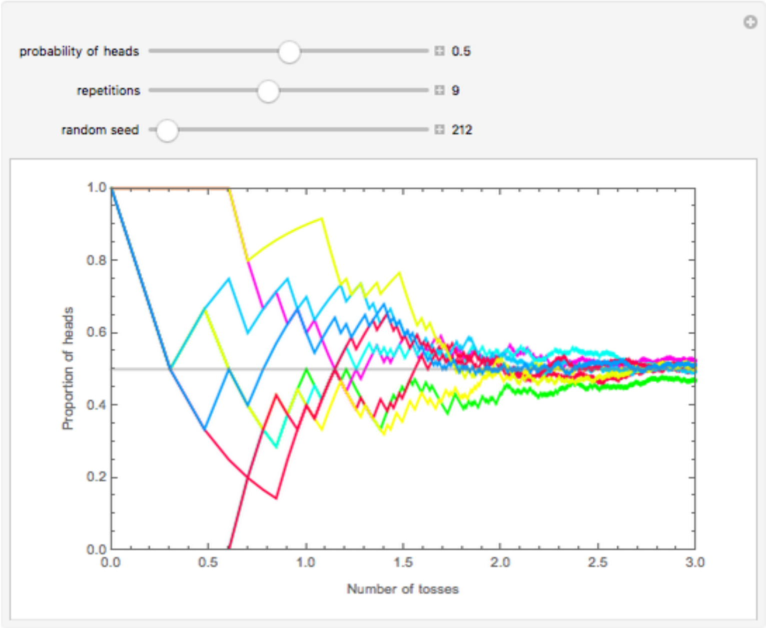
Diagonals: Scalar Variance

Estimation Theory

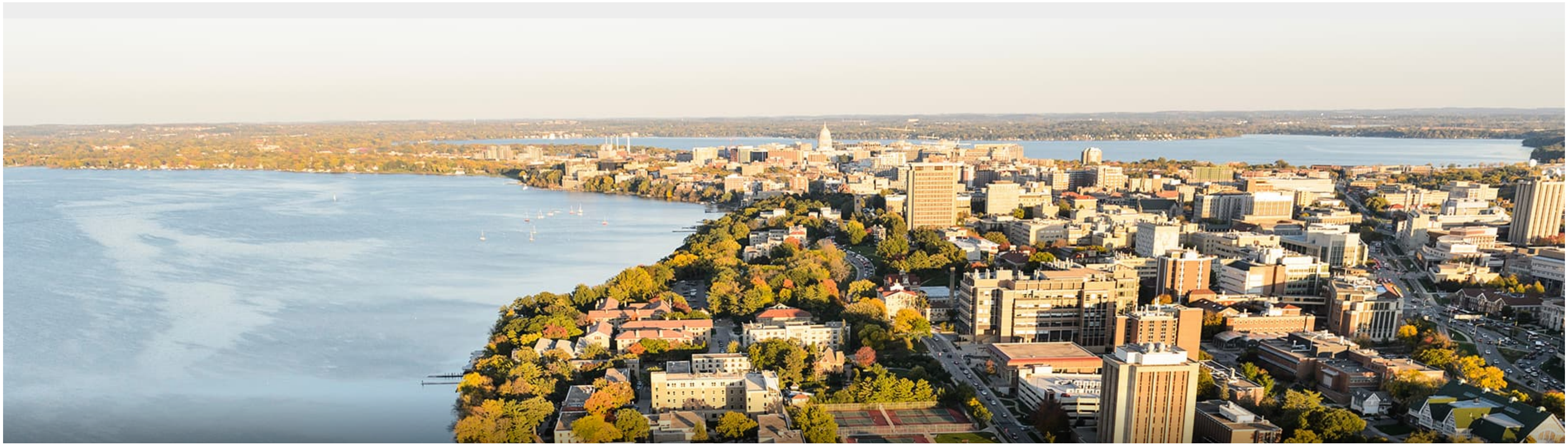
- How do we know that the sample mean is a good estimate of the true mean?
  - Concentration inequalities

$$P(|\mathbb{E}[X] - \hat{\mathbb{E}}[X]| \geq t) \leq \exp(-2nt^2)$$

  - Law of large numbers
  - Central limit theorems, etc.



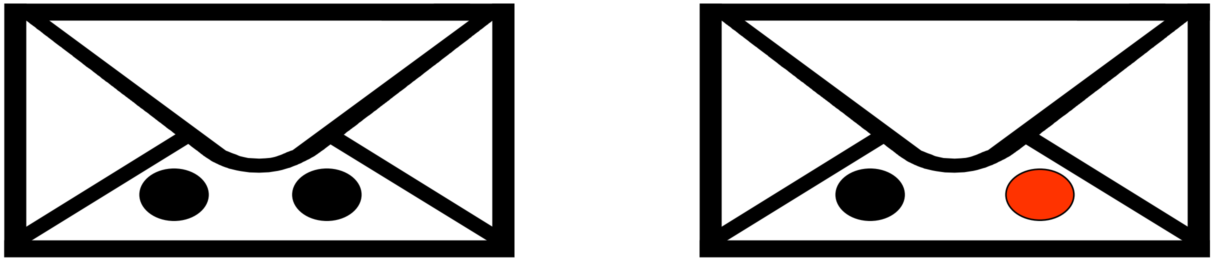
Wolfram Demo



Break & Quiz

Two Envelopes Problem

- We have two envelopes:
  - E_1 has two black balls, E_2 has one black, one red
  - The **red** one is worth $100. Others, zero
  - Open an envelope, see one ball. Then, can switch (or not).
  - You see a black ball. **Switch?**



Two Envelopes Solution

- Let's solve it.

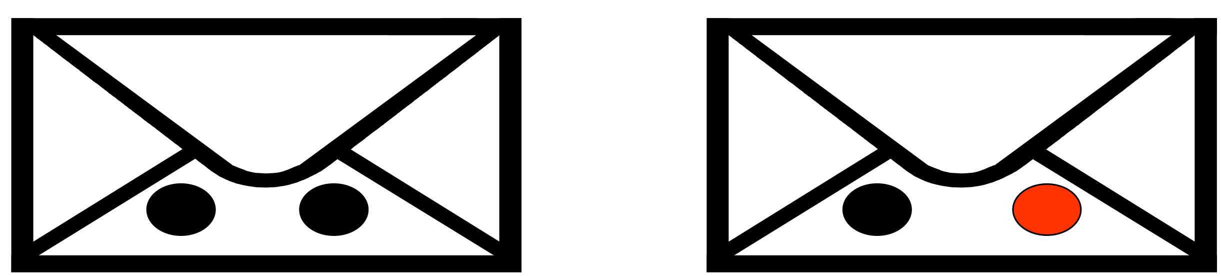
$$P(E_1|\text{Black ball}) = \frac{P(\text{Black ball}|E_1)P(E_1)}{P(\text{Black ball})}$$

- Now plug in:

$$P(E_1|\text{Black ball}) = \frac{1 \times \frac{1}{2}}{P(\text{Black ball})}$$

$$P(E_2|\text{Black ball}) = \frac{\frac{1}{2} \times \frac{1}{2}}{P(\text{Black ball})}$$

So switch!



Break & Quiz

Q 3.1: 50% of emails are spam. Software has been applied to filter spam. A certain brand of software can detect 99% of spam emails, and the probability for a false positive (a non-spam email detected as spam) is 5%. Now if an email is detected as spam, then what is the probability that it is in fact a nonspam email?

A. 5/104
B. 95/100
C. 1/100
D. 1/2

Break & Quiz

Q 3.1: 50% of emails are spam. Software has been applied to filter spam. A certain brand of software claims that it can detect 99% of spam emails, and the probability for a false positive (a non-spam email detected as spam) is 5%. Now if an email is detected as spam, then what is the probability that it is in fact a nonspam email?

A. **5/104**
B. 95/100
C. 1/100
D. 1/2

Outline

- **Review, SVMs, Kernels**
  - Duality, feature maps, kernel trick
- **Probability Tutorial**
  - Basics, joint probability, conditional probabilities, etc
- **Bayesian Networks**
  - Definition, examples, inference

Bayesian Networks Example

- Consider the following 5 binary random variables:

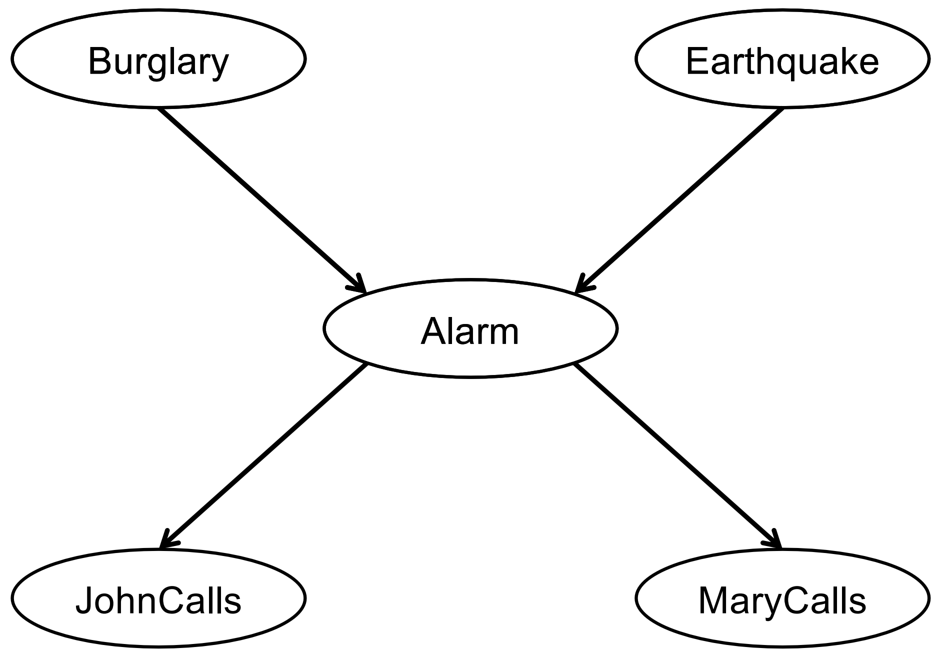
  B = a burglary occurs at the house
  E = an earthquake occurs at the house
  A = the alarm goes off
  J = John calls to report the alarm
  M = Mary calls to report the alarm

- Suppose Burglary or Earthquake can trigger Alarm, and Alarm can trigger John's call or Mary's call

- Now we want to answer queries like what is $P(B \mid M, J)$?

Bayesian Networks Example

- Set up a network that shows how random variables influence others:

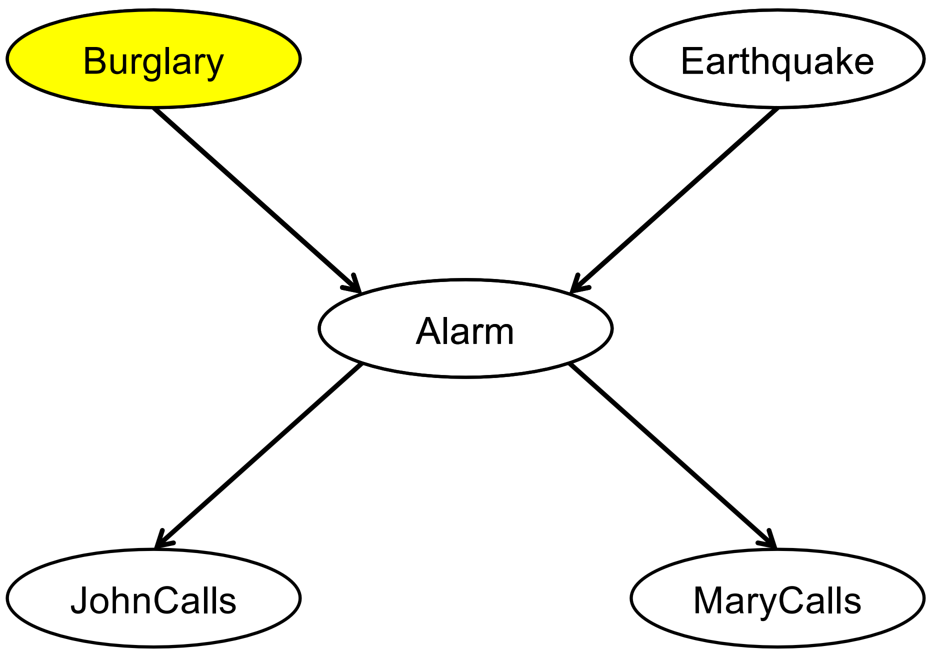


Bayesian Networks Example

- Set up a network that shows how random variables influence others:

$P(B)$

t	f
0.001	0.999

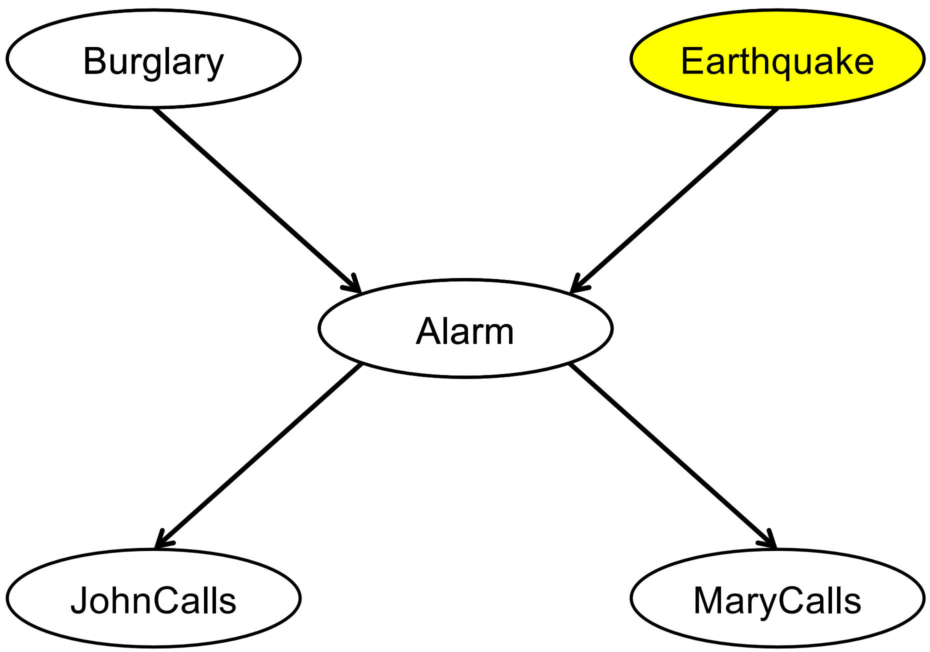


Bayesian Networks Example

- Set up a network that shows how random variables influence others:

P (*B*)

t	f
0.001	0.999



P (*E*)

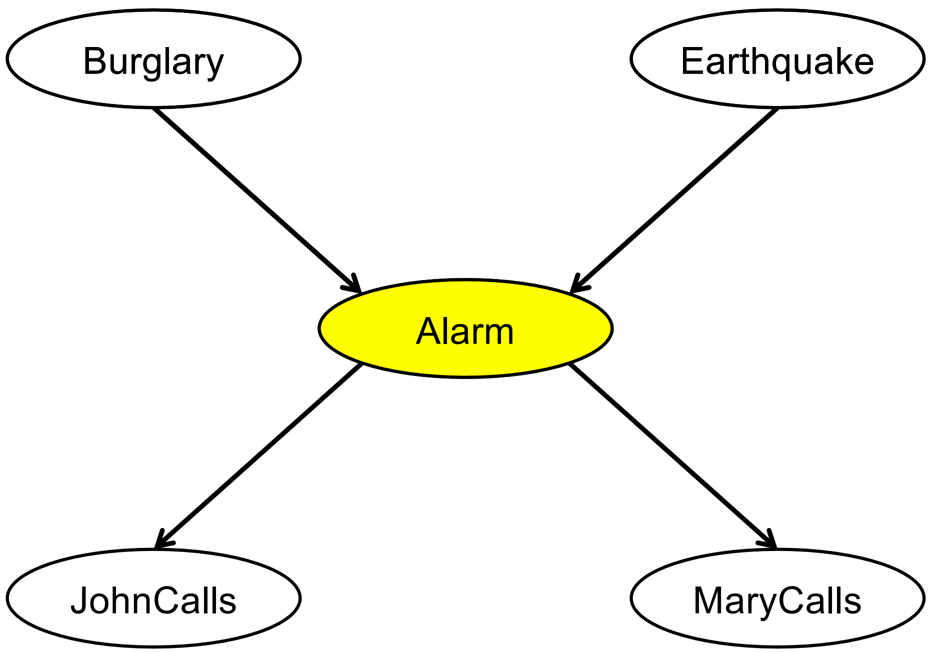
t	f
0.001	0.999

Bayesian Networks Example

- Set up a network that shows how random variables influence others:

P (B)

t	f
0.001	0.999



P (E)

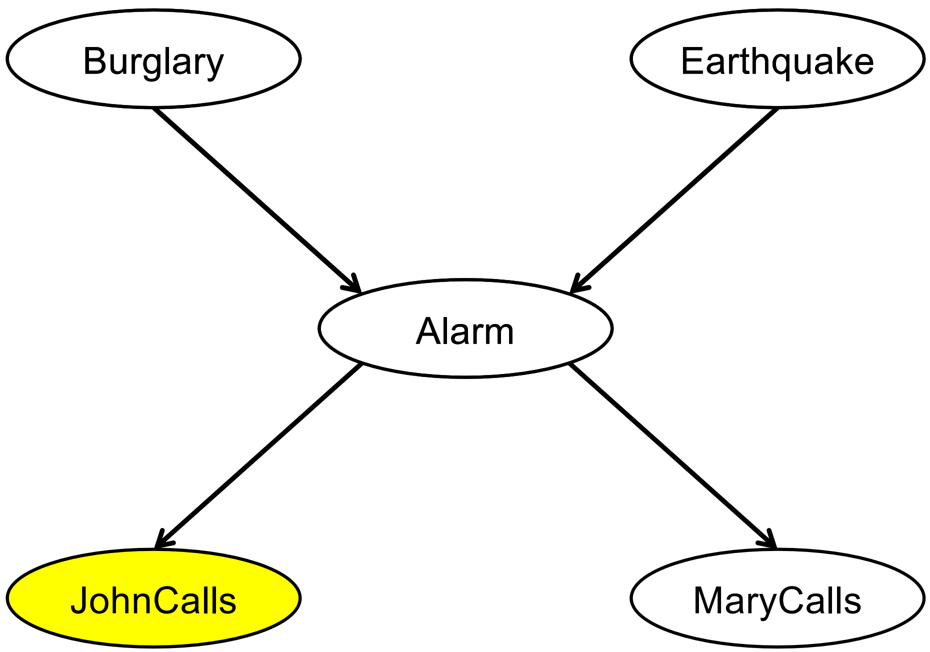
t	f
0.001	0.999

P (A | B, E)

B	*E*	t	f
t	t	0.95	0.05
t	f	0.94	0.06
f	t	0.29	0.71
f	f	0.001	0.999

Bayesian Networks Example

- Set up a network that shows how random variables influence others:



P (*B*)

t	f
0.001	0.999

P (*E*)

t	f
0.001	0.999

P (*A* | *B, E*)

B	*E*	t	f
t	t	0.95	0.05
t	f	0.94	0.06
f	t	0.29	0.71
f	f	0.001	0.999

P (*J* | *A*)

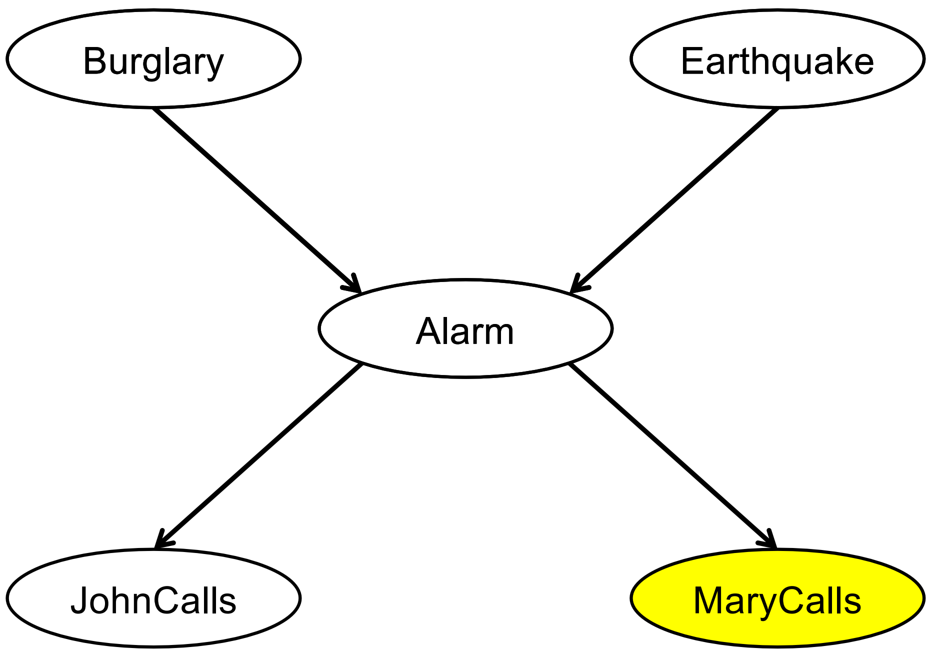
A	t	f
t	0.9	0.1
f	0.05	0.95

Bayesian Networks Example

- Set up a network that shows how random variables influence others:

P (B)

t	f
0.001	0.999



P (E)

t	f
0.001	0.999

P (A | B, E)

B	*E*	t	f
t	t	0.95	0.05
t	f	0.94	0.06
f	t	0.29	0.71
f	f	0.001	0.999

P (J | A)

A	t	f
t	0.9	0.1
f	0.05	0.95

P (M | A)

A	t	f
t	0.7	0.3
f	0.01	0.99

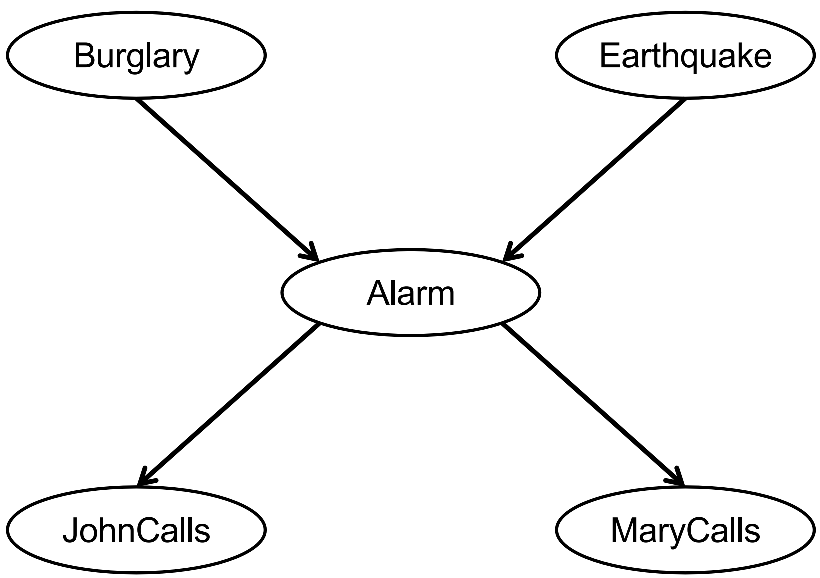
Bayesian Networks: Definition

- A BN consists of a **Directed Acyclic Graph (DAG**) and a set of **conditional probability distribution**s
- The DAG:
  - each node denotes a random variable
  - each edge from X to Y represents that X *directly influences* Y
  - (formally: each variable X is independent of its non-descendants given its parents)
  - **Each CPD: represents** $P(X \mid Parents(X))$

$$p(x_1, \ldots, x_d) = \prod_{v \in V} p(x_v | x_{\mathrm{pa}(v)})$$

Bayesian Networks: Parameter Counting

- Parameter reduction: a standard representation of the joint distribution for the Alarm example has $2^5 = 32$ parameters
- the BN representation of this distribution has 20 parameters



$$
\begin{aligned}
&P(B,E,A,J,M) \\
&= P(B) \\
&\times P(E) \\
&\times P(A \mid B,E) \\
&\times P(J \mid A) \\
&\times P(M \mid A)
\end{aligned}
$$

Inference in Bayesian Networks

Given: values for some variables in the network (*evidence*), and a set of ***query*** variables

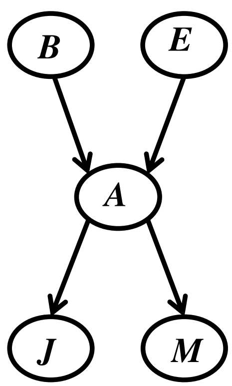
Do: compute the posterior distribution over the query variables

- variables that are neither evidence variables nor query variables are ***hidden*** variables
- the BN representation is flexible enough that any set can be the evidence variables and any set can be the query variables

Inference by Enumeration

- Let a denote A=true, and $\neg a$ denote A=false
- Suppose we're given the query: $P(b \mid j, m)$

  "probability the house is being burglarized given that John and Mary both called"
- From the graph structure we can first compute:



$$P(b, j, m) = \sum_{e,\neg e} \sum_{a,\neg a} P(b) P(E) P(A \mid b, E) P(j \mid A) P(m \mid A)$$

sum over possible values for E and A variables (e, $\neg e$, a, $\neg a$)

Inference by Enumeration

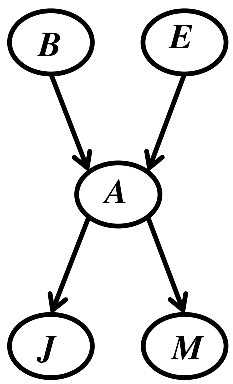
$$P(b, j, m) = \sum_{e,\neg e}\sum_{a,\neg a} P(b)P(E)P(A \mid b, E)P(j \mid A)P(m \mid A)$$

$$= P(b)\sum_{e,\neg e}\sum_{a,\neg a} P(E)P(A \mid b, E)P(j \mid A)P(m \mid A)$$

P(B)
0.001

P(E)
0.001

B	*E*	*P(A)*
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.00 1



A	*P(J)*
t	0.9
f	0.05

A	*P(M)*
t	0.7
f	0.01

$B \quad E \quad A \quad J \quad M$

$$= 0.001 \times (0.001 \times 0.95 \times 0.9 \times 0.7 + \qquad e, a$$

$$0.001 \times 0.05 \times 0.05 \times 0.01 + \qquad e, \neg a$$

$$0.999 \times 0.94 \times 0.9 \times 0.7 + \qquad \neg e, a$$

$$0.999 \times 0.06 \times 0.05 \times 0.01) \qquad \neg e, \neg a$$

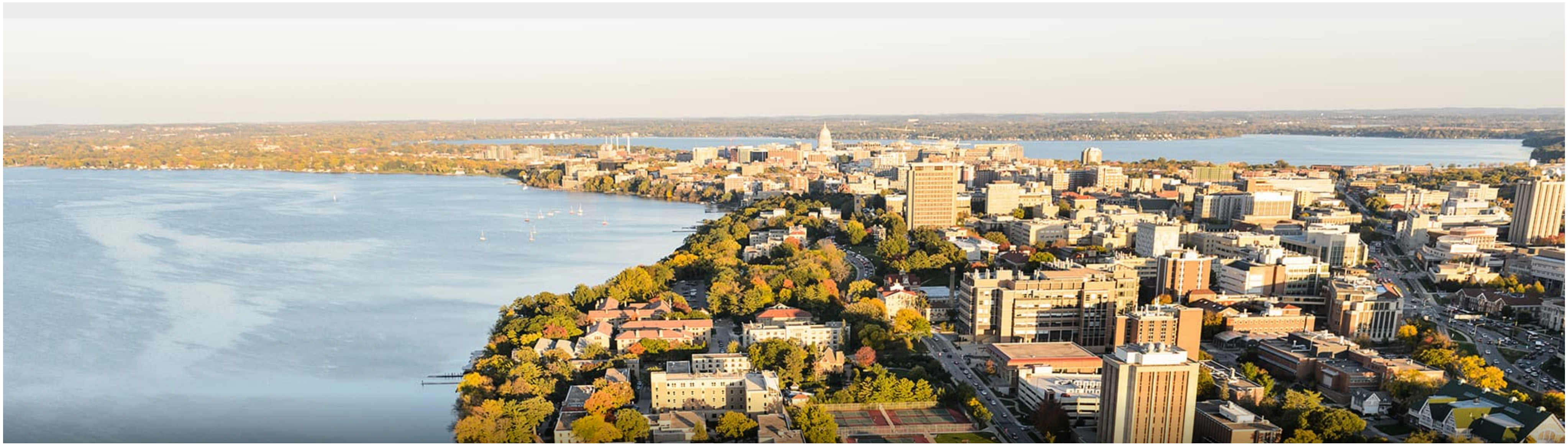
Inference by Enumeration

- Next do equivalent calculation for $P(\neg b,\ j, m)$ and determine $P(b \mid j, m)$

$$P(b \mid j,m) = \frac{P(b, j,m)}{P(j,m)} = \frac{P(b, j,m)}{P(b, j,m) + P(\neg b, j,m)}$$

So: exact method, but can be intractably hard.

- Some cases: efficient
- Approximate inference sometimes available



Thanks Everyone!

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