Learning Gaussian Covariance Robustly

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August 14th, 2018
Yesterday, we discussed how to learn the mean of a Gaussian with known covariance matrix even under adversarial noise. Today we will discuss how to learn the covariance matrix.
Outline

- Problem Setup
- Rough Estimates
- Refined Estimates
- Mean and Covariance
Basic Problem

- Consider $G = \mathcal{N}(\mu, \Sigma) \subset \mathbb{R}^n$.
- Given $N$ samples, $\epsilon$-fraction adversarially corrupted.
- Learn approximations to $\mu, \Sigma$. 
Known Mean

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By translation, we can assume that $\mu = 0$. 
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where

$$\|A\|_F^2 = \sum_{i,j} A_{i,j}^2.$$
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$$\|A\|_F^2 = \sum_{i,j} A_{i,j}^2.$$

Hope get estimate $\hat{\Sigma}$ so that:

$$\|\Sigma^{-1/2}\hat{\Sigma}\Sigma^{-1/2} - I\|_F = \tilde{O}(\epsilon).$$
Basic Technique

Learning the mean of a Gaussian is equivalent to

- Learning $\mathbb{E}[L(G)]$ for degree-1 polynomials $L$.

- Learning the first moments of $G$. 

Learning the covariance of a mean 0 Gaussian is equivalent to:

- Learning $\mathbb{E}[p(G)]$ for even, degree-2 polynomials $p$.

- Learning $\mathbb{E}[GG^T]$.

We will use the last of these formulations.
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We will use the last of these formulations.
We have reduced the problem to robustly estimating the mean of the $n^2$-dimensional random variable $X = GG^T$. Since $\text{Cov}(G) = \Sigma = \mathbb{E}[X]$. 

Let $\Sigma = \text{Cov}(X)$. If $\Sigma \ll I_{n^2}$, can learn $X$ to $L_2$ error (and thus, $\Sigma$ to Frobenius error) $O(\sqrt{\epsilon})$. So, what is $\Sigma$?
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Robust Mean Estimation

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So, what is \( \Sigma \)?
Suppose that $y_i$ are an orthonormal basis of linear functions of $G$.
- $\text{Cov}(y_i, y_j) = \delta_{i,j}$
Computing $\Sigma$

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- $y_iy_j (i \neq j)$ and $(y_i^2 - 1)/\sqrt{2}$ form an orthonormal basis for even degree-2 polynomials of $G$. 

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- For matrix $A$,

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$$= 2 \left\| \Sigma^{1/2} \left( \frac{A + A^T}{2} \right) \Sigma^{1/2} \right\|_F^2.$$
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So, for example, if $\Sigma \leq I$, $\Sigma \ll I$. 
Bootstrapping

- To learn $\Sigma$, need to learn $\mathbb{E}[X]$ robustly.
- Can learn $\mathbb{E}[X]$ robustly, if we have an upper bound on $\Sigma$.
- Can find $\Sigma$ if we know $\Sigma$. 
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Bootstrap better and better approximations to $\Sigma$!
Critical Point: If $\Sigma \leq \Sigma_0$, then $\Sigma \leq \Sigma_0$, i.e.

$$2 \left\| \Sigma^{1/2} \left( \frac{A + A^T}{2} \right) \Sigma^{1/2} \right\|_F^2 \leq 2 \left\| \Sigma_0^{1/2} \left( \frac{A + A^T}{2} \right) \Sigma_0^{1/2} \right\|_F^2$$

for all $A$. 
Upper Bounds

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for all $A$.

So if $\Sigma \leq \Sigma_0$, then $\text{Cov}(\Sigma_0^{-1/2} X \Sigma_0^{-1/2}) \ll I_{n^2}$. Can get estimate $\hat{\Sigma}$ with

$$\left\| \Sigma_0^{-1/2} \left( \hat{\Sigma} - \Sigma \right) \Sigma_0^{-1/2} \right\|_F = O(\sqrt{\epsilon}).$$
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So $\hat{\Sigma} = \Sigma + O(\sqrt{\epsilon})\Sigma_0$. 
Iteration

- Start with some upper bound $\Sigma_0 \geq \Sigma$ (twice the sample covariance works with high probability).
- Get approximation $\hat{\Sigma}_0$.
- Use $\Sigma_1 = \hat{\Sigma}_0 + C \sqrt{\epsilon} \Sigma_0$ as new upper bound.
- Get approximation $\hat{\Sigma}_1$.
- Use $\Sigma_2 = \hat{\Sigma}_1 + C \sqrt{\epsilon} \Sigma_1$ as new upper bound.

... 

- Have $\Sigma_{i+1} \leq \Sigma + O(\sqrt{\epsilon}) \Sigma_i$. Eventually get $\Sigma_\infty \leq \Sigma(1 + O(\sqrt{\epsilon}))$, and $\hat{\Sigma}$ with $\|\Sigma^{-1/2} \hat{\Sigma} \Sigma^{-1/2} - I\|_F = O(\sqrt{\epsilon})$. 

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Error Idea

- Have error $O(\sqrt{\epsilon})$.
  - Best possible using only bounds on $\text{Cov}(X)$. 

Simplifying Assumption: $\Sigma \approx I$. 

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**Simplifying Assumption:** $\Sigma \approx I$. 
**Standard Result:** If $p$ is a degree-2 polynomial with $\text{Var}(p(G)) = O(1)$, then
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Therefore, $X$ has exponential concentration about its mean in any direction.
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$\text{Cov}($Good Samples$) = C + O(\delta + \epsilon \log^2(1/\epsilon))$.

If Sample covariance at most $C + O(\delta + \epsilon \log^2(1/\epsilon))$, then sample mean accurate to error $O(\sqrt{\epsilon\delta} + \epsilon \log(1/\epsilon))$. 

Algorithm

To approximate $\mathbb{E}[X]$:

1. Compute sample covariance matrix $\hat{C}$
2. Find largest eigenvalue of $\hat{C} - C$
   - If none, larger than $O(\delta + \epsilon \log^2(1/\epsilon))$, return sample mean.
3. Otherwise, eigenvector $v$ with large eigenvalue.
   - Variance in that direction is more than $\epsilon \log^2(1/\epsilon)$ larger than it should be due to $O(\epsilon)$-fraction of errors.
   - Most of these errors at distance much more than $\log(1/\epsilon)$ from mean.
   - Few good samples this far out.
   - Create filter.
4. Apply filter to samples and return to step 1.
If we had an approximation to $\Sigma$ with error $O(\delta)$, can obtain one with error $O(\sqrt{\delta \epsilon} + \epsilon \log(1/\epsilon))$. 

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Today: If $\mu = 0$, robustly learn $\Sigma$. 

Question: What if neither $\Sigma$ nor $\mu$ is known?
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Question: What if neither $\Sigma$ nor $\mu$ is known?
Consider differences of pairs of samples $G_{2i} - G_{2i+1}$. 

\[ \text{Distributed as } N(0, 2\Sigma), \text{ with } 2\epsilon \text{ error.} \]

\[ \text{Use to learn } \hat{\Sigma}, \text{ an approximation to } \Sigma \text{ with } O(\epsilon \log(1/\epsilon)) \text{ error.} \]

\[ \hat{\Sigma}^{-1/2} G \approx N(\hat{\Sigma}^{-1/2} \mu, I) \]

\[ \text{Treat difference as } O(\epsilon \log(1/\epsilon)) \text{ adversarial error.} \]

\[ \text{Use to learn approximation to } \mu. \]

Final result: Learn distribution for $G$ to $\tilde{O}(\epsilon)$ error in total variational distance.
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Trick

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Conclusion

We can learn the mean and covariance of an unknown Gaussian robustly. In order to do so, we need to consider the 2nd and 4th moments of the distribution in question. Today we will look into cases where even higher moments are useful.