Robust estimation via (non)convex M-estimation

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Workshop on computational efficiency and
high-dimensional robust statistics
TTI Chicago

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Outline

1. Regularized M-estimators
   - Statistical M-estimation
   - Nonconvexity
   - Consistency of local optima

2. High-dimensional robust regression
   - Statistical consistency
   - Asymptotic normality
   - Two-step M-estimators

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- **Prediction/regression problem:** Observe $\{(x_i, y_i)\}_{i=1}^n$, estimate

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- Statistical M-estimator:

$$\widehat{\beta} \in \arg\min_{\beta} \left\{ \frac{1}{n} \sum_{i=1}^n \ell(\beta; x_i, y_i) \right\}$$

in high dimensions, may be ill-conditioned, large solution space

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- High-dimensional regularized M-estimator:

$$\widehat{\beta}_{\text{Lasso}} \in \arg\min_{\beta} \left\{ \frac{1}{n} \sum_{i=1}^n (y_i - x_i^T \beta)^2 + \lambda \|\beta\|_1 \right\}$$

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- OLS estimator

$$\widehat{\beta} \in \arg\min_{\beta} \left\{ \frac{1}{n} \sum_{i=1}^n (z_i^T \beta - y_i)^2 + \lambda \|\beta\|_1 \right\}$$

  statistically inconsistent

Corrected losses

- L. & Wainwright '12 propose natural method for correcting loss for linear regression:

$$\widehat{\beta}_{\text{OLS}} \in \arg\min_{\beta} \left\{ \frac{1}{2} \beta^T \frac{X^T X}{n} \beta - \frac{y X^T}{n} \beta + \rho_\lambda(\beta) \right\}$$

$$\widehat{\beta}_{\text{corr}} \in \arg\min_{\beta} \left\{ \frac{1}{2} \beta^T \widehat{\Gamma} \beta - \widehat{\gamma}^T \beta + \rho_\lambda(\beta) \right\}$$

$(\widehat{\Gamma}, \widehat{\gamma})$ estimators for $(\text{Cov}(x_i), \text{Cov}(x_i, y_i))$ based on $\{(z_i, y_i)\}_{i=1}^n$

Example: Additive noise

- Additive noise: $Z = X + W$, use

$$\widehat{\Gamma} = \frac{Z^T Z}{n} - \Sigma_w, \qquad \widehat{\gamma} = \frac{Z^T y}{n}$$

- However, corrected objective **nonconvex**:

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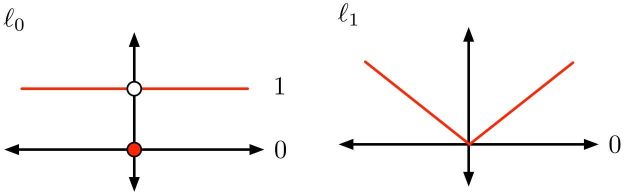
- Fortunately, local optima have good properties

Nonconvex regularizers

- ℓ_1 is "convexified" version of ℓ_0



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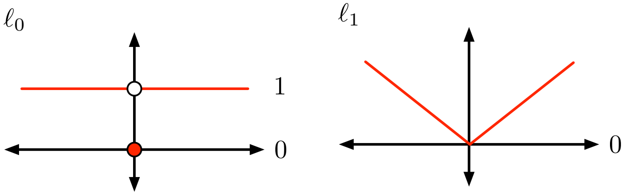
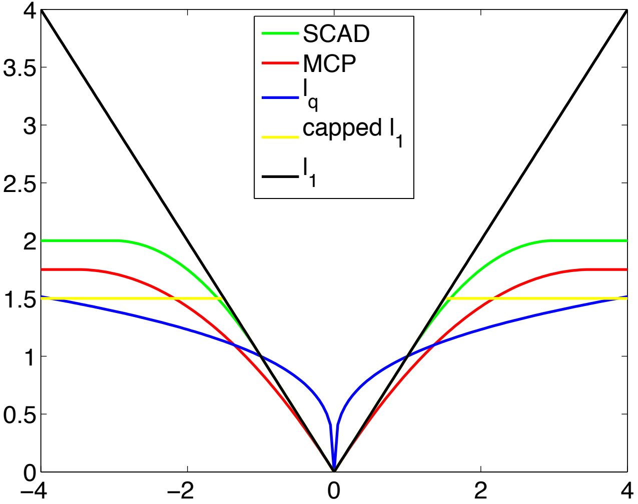


- **But** ℓ_1 penalizes larger coefficients more, causes *solution bias*

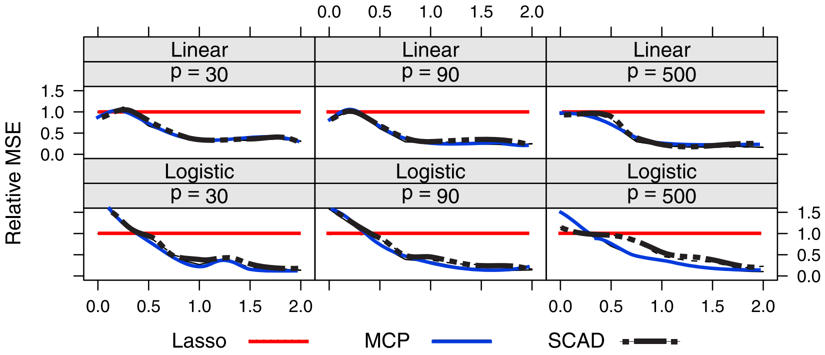
Alternative regularizers

- Various nonconvex regularizers in literature (Fan & Li '01, Zhang '10, etc.)



Empirical benefits

- Nonconvex regularizers show **significant improvement** (Breheny & Huang '11)



Local vs. global optima

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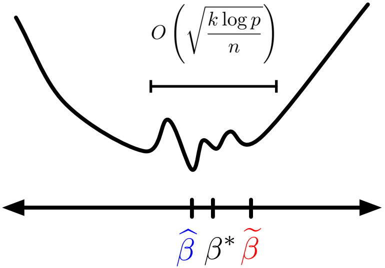

- **L. & Wainwright '13:** All stationary points of $\mathcal{L}_n(\beta) + \rho_\lambda(\beta)$ close when nonconvexity smaller than curvature

Measures of closeness

- Various measures of statistical consistency

$$\widehat{\beta} \in \arg\min_{\beta} \left\{ \frac{1}{n} \sum_{i=1}^{n} \ell(\beta; x_i, y_i) + \rho_\lambda(\beta) \right\}$$

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- **Prediction:** $\frac{1}{n}\sum_{i=1}^{n} \ell(\widehat{\beta}; x_i, y_i) \to 0$
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- Interested in cases where ℓ and ρ_λ possibly *nonconvex*

Estimation/prediction consistency

- Composite objective function

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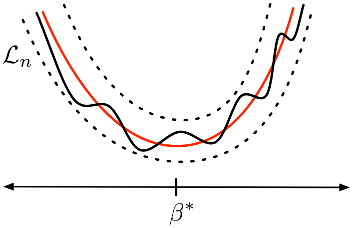
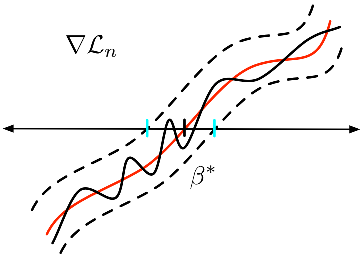
Geometric intuition

- **Population-level** convexity, **finite-sample** nonconvexity

function view



gradient view



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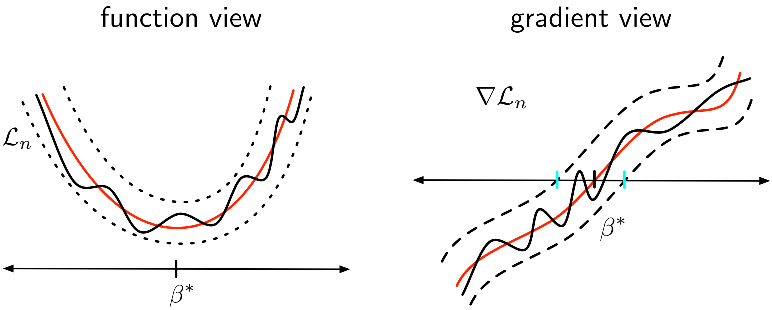
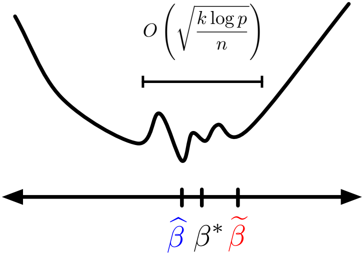


- Population-level objective $\mathcal{L}$ strongly convex, $\alpha > \mu$
- RSC quantifies convergence rate of $\nabla\mathcal{L}_n \longrightarrow \nabla\mathcal{L}$

More formally



- Stationary points statistically indistinguishable from global optima

$$\langle \nabla \mathcal{L}_n(\widetilde{\beta}) + \nabla \rho_\lambda(\widetilde{\beta}),\, \beta - \widetilde{\beta} \rangle \geq 0, \qquad \forall \beta \text{ feasible}$$

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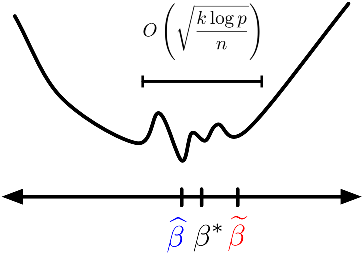

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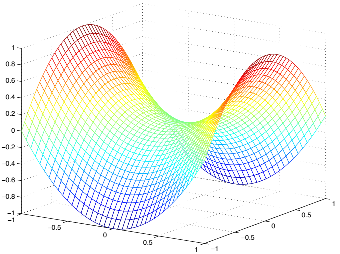
  - Restricted strong convexity of $\mathcal{L}_n$
  - Bound on nonconvexity of ρ_λ
- "Oracle" result under additional condition on ρ_λ

Conditions on $\mathcal{L}_n$

- **Restricted strong convexity** (Negahban et al. '12):

$$\langle \nabla \mathcal{L}_n(\beta^* + \Delta) - \nabla \mathcal{L}_n(\beta^*),\ \Delta \rangle \geq \begin{cases} \alpha \|\Delta\|_2^2 - \tau \frac{\log p}{n} \|\Delta\|_1^2, & \forall \|\Delta\|_2 \leq r \\ \alpha \|\Delta\|_2 - \tau \sqrt{\frac{\log p}{n}} \|\Delta\|_1, & \text{o.w.} \end{cases}$$



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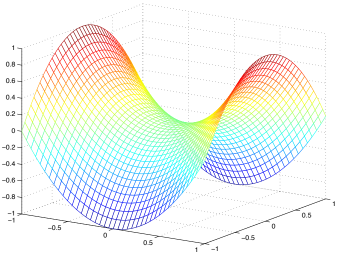

- Holds for various convex/nonconvex losses:
  - OLS & corrected OLS for linear regression, log likelihood for GLMs
  - Huber loss for robust regression

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    - Neither: capped-ℓ_1, bridge penalty ($\ell_q, 0 < q < 1$)

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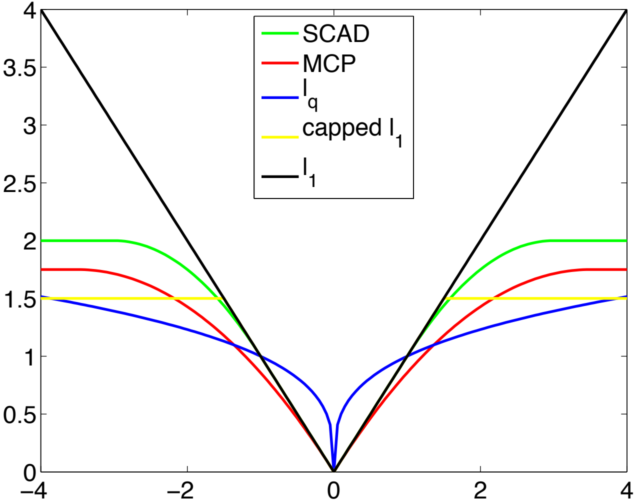

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Theorem (L. & Wainwright '13)

Suppose R is chosen s.t. β^* is feasible, and λ satisfies

$$\max\left\{ \|\nabla \mathcal{L}_n(\beta^*)\|_\infty, \; \alpha\sqrt{\frac{\log p}{n}} \right\} \precsim \lambda \precsim \frac{\alpha}{R}.$$

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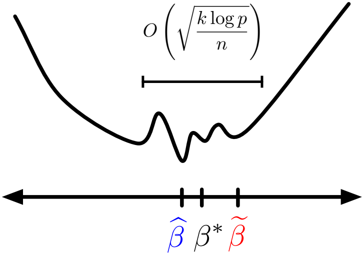
For $n \ge \frac{C\tau^2}{\alpha^2} R^2 \log p$, **any stationary point** $\widetilde{\beta}$ *satisfies*

$$\|\widetilde{\beta} - \beta^*\|_2 \precsim \frac{\lambda\sqrt{k}}{\alpha - \mu}, \qquad \text{where } k = \|\beta^*\|_0.$$

More formally



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- Use M-estimator

$$\widehat{\beta} \in \arg\min_{\beta} \left\{ \frac{1}{n} \sum_{i=1}^{n} \ell(x_i^T \beta - y_i) \right\}$$

Classes of loss functions

- **Bounded** ℓ' limits influence of outliers:

$$IF((x,y); T, F) = \lim_{t \to 0^+} \frac{T((1-t)F + t\delta_{(x,y)}) - T(F)}{t}$$
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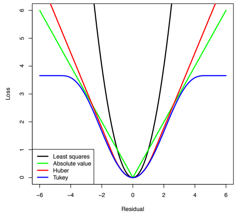

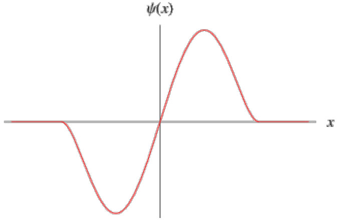

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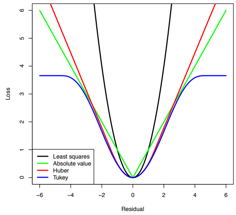

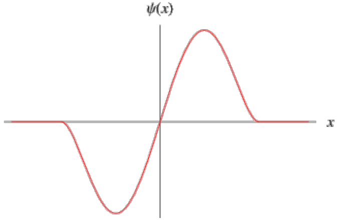

- But bad for optimization!!

High-dimensional M-estimators

- **Natural idea:** For $p > n$, use regularized version:

$$\widehat{\beta} \in \arg\min_{\beta} \left\{ \frac{1}{n} \sum_{i=1}^{n} \ell(x_i^T \beta - y_i) + \lambda \|\beta\|_1 \right\}$$

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Complications:

- Optimization for nonconvex ℓ?
- Statistical theory? Are certain losses provably better than others?

Overview of results

- When $\|\ell'\|_\infty < C$, **global optima** of high-dimensional M-estimator satisfy

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- Local optima may be obtained via **two-step algorithm**

Theoretical insight

- Lasso analysis (e.g., van de Geer '07, Bickel et al. '08):

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- Sub-Gaussian assumptions on x_i's and ϵ_i's provide $\mathcal{O}\left(\sqrt{\frac{k \log p}{n}}\right)$ bounds, minimax optimal

Theoretical insight

- **Key observation:** For general loss function, if $\lambda \geq 2 \left\| \frac{X^T \ell'(\epsilon)}{n} \right\|_\infty$, obtain
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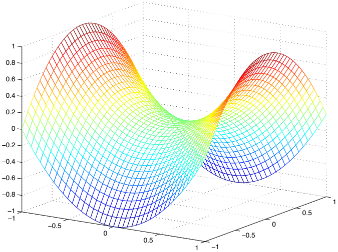
- $\ell'(\epsilon)$ sub-Gaussian whenever ℓ' bounded
$\implies$ can achieve estimation error
$$\|\widehat{\beta} - \beta^*\|_2 \leq c\sqrt{\frac{k \log p}{n}},$$
without assuming ϵ_i is sub-Gaussian

Local statistical consistency

- **Local** RSC condition: For $\Delta := \beta_1 - \beta_2$,

$$\langle \nabla \mathcal{L}_n(\beta_1) - \nabla \mathcal{L}_n(\beta_2),\ \Delta \rangle \geq \alpha \|\Delta\|_2^2 - \tau \frac{\log p}{n} \|\Delta\|_1^2, \quad \forall \|\beta_j - \beta^*\|_2 \leq r$$



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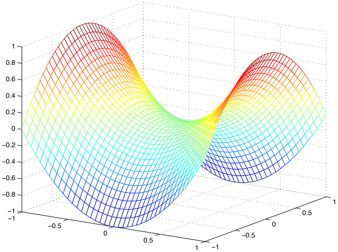
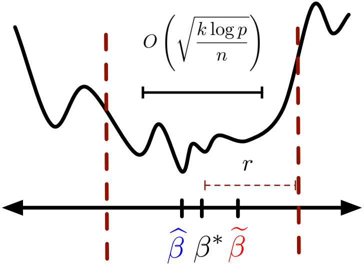


- Only requires restricted curvature within constant-radius region around β^*

Consistency of local stationary points



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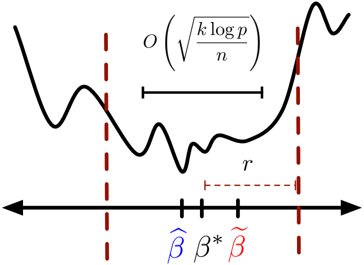

Theorem (L. '15)

Suppose $\mathcal{L}_n$ satisfies α-local RSC and ρ_λ is μ-amenable, with $\alpha > \mu$. Suppose (λ, R) are chosen appropriately. For $n \gtrsim \frac{\tau}{\alpha-\mu} k \log p$, **any stationary point** $\widetilde{\beta}$ **s.t.** $\|\widetilde{\beta} - \beta^*\|_2 \leq r$ *satisfies*

$$\|\widetilde{\beta} - \beta^*\|_2 \lesssim \frac{\lambda\sqrt{k}}{\alpha - \mu}.$$

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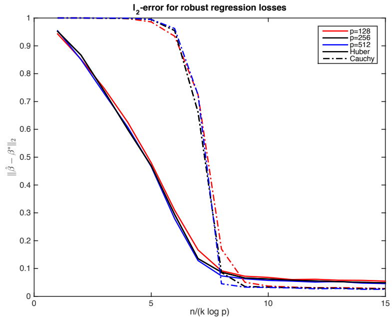
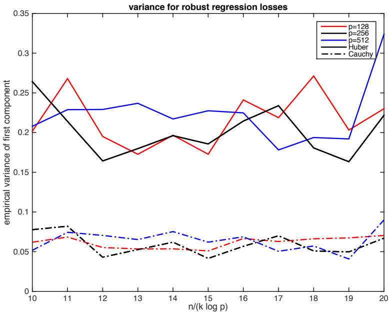
- In low-dimensional settings, MLE maximally efficient *with respect to variance*
- Although MLE may behave erratically when $\frac{p}{n} \to (0, 1]$, can achieve simple asymptotic normality results under *sparsity* assumption, via oracle property

Asymptotic efficiency





- ℓ_2-error and empirical variance of M-estimators when errors follow Cauchy distribution (SCAD regularizer)

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Descending ψ-functions are tricky, especially when the starting values for the iterations are non-robust. . . . It is therefore preferable to start with a monotone ψ, iterate to death, and then append a few (1 or 2) iterations with the nonmonotone ψ. — Huber 1981, pp. 191–192

Two-step algorithm (L. 15)

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- Theoretical guarantees on (rate of) convergence to optimal point

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- Output is **computationally** and **statistically** efficient:

- Computational guarantee on rate of convergence in each step of *M*-estimation algorithm
- Statistical guarantee on asymptotic efficiency of estimator (assuming β-min condition and (μ, γ)-amenability)

Summary

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  - Global RSC condition $\implies$ *all* stationary points within statistical error of β^*
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  - Oracle estimator with (μ, γ)-amenable regularizer $\Longrightarrow$ asymptotic efficiency
  - Two-step *M*-estimator produces local oracle solutions

References

- **P. Loh** and M. J. Wainwright (2015). Regularized M-estimators with nonconvexity: Statistical and algorithmic theory for local optima. *Journal of Machine Learning Research*.
- **P. Loh** (2017). Statistical consistency and asymptotic normality for high-dimensional robust M-estimators. *Annals of Statistics.*

Thank you!